

Theory of weights in p -adic cohomology

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Abstract

Let k be a finite field of characteristic $p > 0$. We construct a theory of weights for overholonomic complexes of arithmetic \mathcal{D} -modules with Frobenius structure on varieties over k . The notion of weight behave like Deligne's one in the ℓ -adic framework: first, the six operations preserve weights, and secondly, the intermediate extension of an immersion preserves pure complexes and weights.

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Introduction

Using Grothendieck's theory of ℓ -adic étale cohomology of varieties over a field of characteristic $p \neq \ell$, Deligne proved Weil's conjectures on the numbers of points of algebraic varieties over finite fields. Moreover, he built a theory of mixedness and weights for constructible ℓ -adic sheaves which is compatible with the "six operations formalism" of the ℓ -adic cohomology, namely the mixedness and weight are stable under the operations $f_!$, f_* , f^* , $f^!$, \otimes and $\mathcal{H}om$ (see [Del80]). Later, using the theory of perverse sheaves, in

[BBD82] Gabber proved the stability of purity and mixedness under intermediate extensions, with which the theory of weights for ℓ -adic cohomology may be regarded as complete. However, the problem of obtaining similar results within a p -adic cohomological framework remained opened. After Dwork's p -adic proof of the rationality of zeta functions, a part of Weil's conjectures, it seems natural to expect better computability of zeta functions with a p -adic approach. This kind of computation are of much interest in coding theory and cryptography. In this paper, we build a p -adic theory of weights.

The first attempt to calculate the weights of some p -adic cohomology was made by Katz and Messing in their famous paper [KM74]. Using Deligne's deep results on weights, especially "*Le théorème du pgcd*", they showed that, for proper smooth varieties, the weight of crystalline cohomology is the same as that of ℓ -adic one. It is reasonable to hope that the coefficient theory of weights parallel to ℓ -adic cohomology should exist in the spirit of the *petit camarade conjecture* [Del80, 1.2.10], even though there were many obstacles that prevented the construction of such a theory. After the work of Katz-Messing, efforts were made until Kedlaya finally obtained in [Ked06] the expected estimation of weights of rigid cohomology, a p -adic cohomology constructed by Berthelot. We do not go into more details of the history, and recommend the reader to consult the excellent explanation in the introduction of [Ked06].

But the context of rigid cohomology was still not completely satisfactory, in the sense that this is not "functorial enough", namely we do not have six operations formalism, specially push-forwards as pointed out in [Ked06, 5.3.3]. In this paper, we use systematically Berthelot's arithmetic \mathcal{D} -modules to complete the program of establishing a p -adic theory of weights stable under six operations. In many applications, such a theory should play as important roles as suggested by the classical situations; for example, the theory of intersection cohomology and its purity, the theory of Springer representations, Lafforgue's proof of Langlands correspondence, etc. In the final part of the paper, we show one such application that the Hasse-Weil L -function for a function field defined by means of ℓ -adic methods and p -adic methods coincide.

Now let us explain our results in more details. Let \mathcal{V} be a complete discretely valued ring of mixed characteristic $(0, p)$, K its field of fractions, and k its residue field which is assumed to be a finite field. In order to obtain a p -adic cohomology on algebraic varieties over k stable under the six operations whose coefficients contain the category of overconvergent isocrystals, Berthelot constructed the theory of arithmetic \mathcal{D} -modules (cf. [Ber02]). Arithmetic \mathcal{D} -modules may be seen as a p -adic analogue of modules over the ring of differential operators over complex varieties or analytic varieties. Berthelot's theory is inspired by his construction of rigid or crystalline cohomology, and we consider not only differential operators of finite order, but also of infinite order subject to a convergence condition. Even though the objects are defined, the preservation of finiteness had been a difficult question. For this, the notion of overholonomicity was introduced by the second author in [Car09b]. Thanks to Kedlaya's semistable reduction theorem of [Ked11], the second author together with Tsuzuki, finally in [CT12], proved that overholonomicity of \mathcal{D} -modules with Frobenius structure is preserved under various cohomological functors.

Using this foundation, in this paper, we go a step further in the development of a good p -adic cohomology, and build a theory of weights for overholonomic complexes with Frobenius structure of arithmetic \mathcal{D} -modules. More precisely, the main result is the following (cf. Theorem 4.1.3, Corollary 4.2.4):

Main Theorem. *Let X be a (realizable) variety over k . We construct the full subcategory $F\text{-}D_{\mathfrak{m}}^b(X/K)$ of $F\text{-}D_{\text{ovhol}}^b(X/K)$ of ι -mixed complexes. The following properties hold:*

(i) *Let $f: X \rightarrow Y$ be a morphism of varieties. The ι -mixedness is stable under functors f_+ , $f_!$, f^+ , $f^!$, \mathbb{D}_X , $\tilde{\otimes}$. More precisely, f_+ and $f^!$ send ι -mixed F -complexes of weight $\geq w$ to those of weight $\geq w$. Similarly, $f_!$ and f^+ send ι -mixed F -complexes of weight $\leq w$ to those of weight $\leq w$, \mathbb{D}_X exchanges ι -mixed F -complexes of weight $\leq w$ to $\geq -w$, and $\tilde{\otimes}$ send ι -mixed F -complexes of weight $(\geq w, \geq w')$ to $\geq w + w'$.*

(ii) *Intermediate extension of an immersion preserves pure F -complexes and weights.*

Concerning part (i), for the convenience of the reader, we recall that, in the very special case where $Y = \text{Spec } k$, the stability under f_+ was already checked by dévissage in overconvergent F -isocrystals from Kedlaya's stability of weights (see [Car06, 8.3.4] ⁽¹⁾). The relative version, namely the stability under push-

⁽¹⁾In fact, the compatibility with Frobenius of the comparison between push-forward and rigid cohomology was implicitly used. A proof of this compatibility is given in [Abe13]

forwards by any morphism of varieties, not only under the structural morphism of X , had been missing: as Kedlaya remarked in [Ked06, 5.3.3], one had at least to work with a theory admitting Grothendieck's six operations, which is only possible in the theory of arithmetic \mathcal{D} -modules. To check part (i), we follow some ideas appearing in Deligne's proof of the stability under push-forwards of [Del80], which leads us to the relative 1 dimensional case. To tackle this relative 1 dimensional case, specially concerning the stability of mixedness, one needs to study thoroughly the monodromy filtration associated to a log convergent isocrystal (this is section 3 of our work). One also notice that any proofs here do not use [Ked06]. Moreover, the part (ii) is completely new in p -adic cohomology theory.

Let us describe the contents of the paper. We restrict our attention to the category of *realizable varieties*, the full subcategory of the category of varieties over k consisting of Y such that there exists an immersion into a proper and smooth formal scheme over \mathcal{V} . We recall that quasi-projective varieties are realizable. In this introduction, let us simply call these objects, varieties.

In §1, we construct the coefficient theory using the foundational results of overholonomic modules. Even though the main ideas had already been provided, with the definition of a new t -structure here, the main theme of the first section is to extend some properties of the theory of arithmetic \mathcal{D} -modules and, at the same time, to make it in a more usable form. Let Y be a variety. First, we recall the construction of the category $F\text{-}D_{\text{ovhol}}^b(Y/K)$ of overholonomic F -complexes over Y/K and the formalism of the six operations for these coefficients. We then introduce a new t -structure on this category whose heart is denoted by $F\text{-}Ovhol(Y/K)$ and which may be seen as the category of arithmetic \mathcal{D} -modules over Y/K (even though they are complexes and not modules). When Y is smooth, we construct a fully faithful subcategory $F\text{-}Isoc^{\dagger\dagger}(Y/K)$ of $F\text{-}Ovhol(Y/K)$ which is equivalent to the category of overconvergent F -isocrystals over Y/K (the latter are the coefficients of the rigid cohomology). In the spirit of the Riemann-Hilbert correspondence, $F\text{-}Ovhol(Y/K)$ (resp. $F\text{-}Isoc^{\dagger\dagger}(Y/K)$) is a p -adic analogue of the category of perverse sheaves (resp. smooth perverse sheaves) over Y . We proceed to check several necessary properties for being six operations formalism. We conclude the section by defining the intermediate extension as in the context of \mathcal{D} -modules over complex varieties. We also check that the analogous properties concerning irreducibility hold in our context. This enables us to define the p -adic intersection cohomology for realizable varieties, which is an advantage of using arithmetic \mathcal{D} -modules.

We fix an isomorphism $\iota: \overline{\mathbb{Q}}_p \cong \mathbb{C}$. In §2, we define the notion of ι -mixedness as Deligne does for the ℓ -adic cohomology in [Del80]. More precisely, in the spirit of the Riemann-Hilbert correspondence, the ι -purity of an object in $F\text{-}Isoc^{\dagger\dagger}(Y/K)$ is by definition a pointwise property (with the natural definition over a point) and the category of ι -mixed objects of $F\text{-}Isoc^{\dagger\dagger}(Y/K)$ is the full subcategory generated by extensions of ι -pure objects. Next, by dévissage in overconvergent F -isocrystals, we extend naturally the notion of ι -mixedness to $F\text{-}D_{\text{ovhol}}^b(Y/K)$. At the end of the section, we estimate the weight of the cohomology on curves, using the methods developed in [AM11] by the first author together with Marmora.

The aim of §3 is to prove the following preliminary results on the ι -mixedness stability. Let X be a proper smooth variety, Z be a strict normal crossing divisor of X and $j: Y := X \setminus Z \rightarrow X$ be the open immersion. Let \mathcal{G} be a log-convergent F -isocrystal on X with logarithmic poles along Z which possesses nilpotent residues and let \mathcal{E} be the object of $F\text{-}Isoc^{\dagger\dagger}(Y/K)$ induced by restriction. If \mathcal{E} is ι -mixed, then we show that $j_!(\mathcal{E})$ is ι -mixed as well. Moreover, when Z is smooth and \mathcal{E} is ι -pure, we prove that the weight of $j_!(\mathcal{E})$ is less than or equal to that of \mathcal{E} . The theorem follows from the study of the monodromy filtration given by the nilpotent residue morphisms of \mathcal{G} .

In the last section §4, we prove the stability of ι -mixedness and weights under the six operations. Roughly speaking, we reduce to the case treated in the previous section by using Kedlaya's semistable reduction theorem (see [Ked11]). Since we need to take special care of the boundary in p -adic cohomology theory compared to the case of ℓ -adic theory, the proof of this theorem is slight involved. Finally, by using Noot-Huyghe's Fourier transform ([NH04]), we establish the stability of the ι -purity under intermediate extension, which is a p -adic analogue of Gabber's purity theorem. We conclude the section with a few applications: the weight filtration for an ι -mixed object of $F\text{-}Ovhol(Y/K)$, the semi-simplicity of ι -mixed F -complexes after forgetting Frobenius structures, and some ℓ -independence type results.

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Notation and convention

Throughout this paper, \mathcal{V} is a complete discrete valuation ring with mixed characteristic $(0, p)$, K is its field of fractions, k is its residue field which is assumed to be perfect. In principle, we denote formal schemes by using script fonts (e.g. \mathcal{X}), and the special fiber is denoted by the corresponding capital letter (e.g. X). A k -variety is a separated scheme of finite type over k . We remark that without loss of generality, one could assume varieties over k are reduced if needed. We often use \star to abbreviate when it is obvious what should be put and do not want to introduce too much notation. If there is no risk of confusion, we sometimes use $(-)$ by meaning respectively. For example " $p^{(\prime)} \cong q^{(\prime)}$ " means " $p \cong q$ (resp. $p' \cong q'$)".

In this paper, we consider one of the following situations:

- (A) Let $q = p^s$ be a power of p , and fix a lifting σ of the s -th Frobenius F of k to \mathcal{V} and K . (§1)
- (B) We are in situation (A), and moreover, we assume that k is a finite field with q elements, and F is the s -th Frobenius. We fix an isomorphism $\iota: \overline{\mathbb{Q}}_p \cong \mathbb{C}$. (Except for §1)

1 Preliminaries on arithmetic \mathcal{D} -modules

Throughout this section, we consider situation (A) in Notation and convention.

1.1 Arithmetic \mathcal{D} -modules over frames and couples

In this subsection, we fix terminologies, and recall the definitions of six functors in the theory of arithmetic \mathcal{D} -modules. Fundamental additional properties are established from the next subsection.

Definition 1.1.1. We define the following categories:

(i) A *frame* (Y, X, \mathcal{P}) is the data consisting of a separated and smooth formal scheme \mathcal{P} over \mathcal{V} , a closed subvariety X of P , an open subscheme Y of X . A morphism of frames $u = (b, a, f): (Y, X, \mathcal{P}) \rightarrow (Y', X', \mathcal{P}')$ is the data consisting of morphisms $b: Y' \rightarrow Y$, $a: X' \rightarrow X$, $f: \mathcal{P}' \rightarrow \mathcal{P}$ such that f induces the other ones.

(ii) An *l.p. frame*⁽²⁾ $(Y, X, \mathcal{P}, \mathcal{Q})$ is the data consisting of a proper and smooth formal scheme \mathcal{Q} over \mathcal{V} , an open formal subscheme \mathcal{P} of \mathcal{Q} , a closed subvariety X of P , an open subscheme Y of X . A morphism of frames $u = (b, a, g, f): (Y, X, \mathcal{P}, \mathcal{Q}) \rightarrow (Y', X', \mathcal{P}', \mathcal{Q}')$ is the data consisting of morphisms $b: Y' \rightarrow Y$, $a: X' \rightarrow X$, $g: \mathcal{P}' \rightarrow \mathcal{P}$, $f: \mathcal{Q}' \rightarrow \mathcal{Q}$ such that f induces the other ones. A morphism of l.p. frames is said to be *complete* if a is proper.

Definition 1.1.2. We define the category of *couples*⁽³⁾ as follows:

- An object (Y, X) is the data consisting of a k -variety X and an open subscheme Y of X such that there exists an l.p. frame of the form $(Y, X, \mathcal{P}, \mathcal{Q})$.

⁽²⁾Abbreviation of "locally proper frame".

⁽³⁾In [Car12a], couples are called "properly realizable couples". Contrary to *ibid.*, we only use this type of couples, so we simplify the name.

- A morphism $u = (b, a): (Y', X') \rightarrow (Y, X)$ of couples is a collection of morphisms of k -varieties $a: X' \rightarrow X$ and $b: Y' \rightarrow Y$ such that b is induced by a . The morphism u is said to be *complete* if a is proper. Let \mathbf{P} be a property of morphisms of schemes. We say⁽⁴⁾ that u is c - \mathbf{P} if u is complete and b satisfies \mathbf{P} . (e.g. u is a c -immersion if u is proper and b is an immersion.)

For a couple (Y, X) , we sometimes denote by \mathbb{Y} using the bold font corresponding to the first data of the couple.

Let $u: \mathbb{Y} \rightarrow \mathbb{Y}'$ be a morphism of couples. By definition there exists a morphism of l.p. frames $(b, a, g, f): (Y, X, \mathcal{P}, \mathcal{Q}) \rightarrow (Y', X', \mathcal{P}', \mathcal{Q}')$ of u , namely a morphism of l.p. frames such that $u = (b, a)$. Moreover, we may take g and f to be smooth. When u is complete, we can even choose g, f to be proper.

Definition 1.1.3. The category of *realizable varieties* is the full subcategory of the category of varieties over k consisting of X such that there exists an immersion into a proper and smooth formal scheme over \mathcal{V} .

1.1.4. Let \mathcal{A} be an additive category endowed with an additive endofunctor $F^*: \mathcal{A} \rightarrow \mathcal{A}$. We define the category $F\text{-}\mathcal{A}$ of F -objects of \mathcal{A} as follows: F -objects consist of (\mathcal{E}, Φ) where \mathcal{E} is an object of \mathcal{A} , and $\Phi: \mathcal{E} \xrightarrow{\sim} F^*\mathcal{E}$ is an isomorphism of \mathcal{E} . Later, we take F to be the Frobenius pull-back, and Φ is called the *Frobenius structure*. A morphism of F -objects is a morphism in \mathcal{A} commuting with Frobenius structures. We have a *faithful* additive functor $\rho: F\text{-}\mathcal{A} \rightarrow \mathcal{A}$ defined by forgetting Frobenius structures. If \mathcal{A} is abelian, $F\text{-}\mathcal{A}$ is abelian as well.

Now, let \mathcal{T} be a triangulated category. Assume given an additive endofunctor F^* of \mathcal{T} . The category $F\text{-}\mathcal{T}$ has a shift functor in an obvious way, and a triangle in $F\text{-}\mathcal{T}$ is said to be *distinguished* if it is distinguished triangle after we forget Frobenius structures. We simply say triangle instead of distinguished triangle for the rest of the paper. We caution, however, that $F\text{-}\mathcal{T}$ is *not* triangulated in general. More precisely, it only satisfies (TR1), (TR2) of [Har66].

Definition 1.1.5. (i). Let (Y, X, \mathcal{P}) be a frame. We define the category $D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ to be the full subcategory of $D_{\text{ovhol}}^b(\mathcal{D}_{\mathcal{P}, \mathbb{Q}}^\dagger)$ consisting of complexes \mathcal{E} such that the canonical homomorphism $\mathcal{E} \rightarrow \mathbb{R}\Gamma_Y^\dagger(\mathcal{E}) := \mathbb{R}\Gamma_X^\dagger \circ (\dagger X \setminus Y)(\mathcal{E})$ is an isomorphism (cf. [Car07, 3.2.1]). The category is triangulated.

(ii). Now, let \mathbb{Y} be a couple (cf. 1.1.2). Take an l.p. frame $(Y, X, \mathcal{P}, \mathcal{Q})$ of \mathbb{Y} . By using Lemma [Car12a, 2.5], we can check that the category $D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ does not depend, up to a canonical equivalence, on the choices of \mathcal{P} and \mathcal{Q} . We denote $D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ by $D_{\text{ovhol}}^b(\mathbb{Y}/K)$. Its objects are called *overholonomic complexes of arithmetic \mathcal{D} -modules on \mathbb{Y}* , or simply *overholonomic complexes on \mathbb{Y}* .

(iii). Finally, let us consider Frobenius structure. On the category $D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$, we have the Frobenius pull-back functor. Thus the results of paragraph 1.1.4 can be applied to get the category $F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$. We also have $F\text{-}D_{\text{ovhol}}^b(\mathbb{Y}/K)$.

Remark. Our category $F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ is an analog of the category of “Weil complexes” [KW01, VI.18], so even though it is not triangulated, it gives us a suitable formalism for the theory of weights.

1.1.6. Let \mathbb{Y} be a couple. Choose an l.p. frame $(Y, X, \mathcal{P}, \mathcal{Q})$ of \mathbb{Y} .

(i) We have the dual functor

$$\mathbb{D}_{Y, \mathcal{P}}: F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}/K) \rightarrow F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}/K).$$

This is defined as follows: we denote by $\mathbb{D}_{\mathcal{P}}$ the $\mathcal{D}_{\mathcal{P}, \mathbb{Q}}^\dagger$ -linear dual (e.g. see [Vir00] or [Ber02]). We define $\mathbb{D}_{Y, \mathcal{P}}$ by $\mathbb{D}_{Y, \mathcal{P}}(\mathcal{E}) := \mathbb{R}\Gamma_Y^\dagger \circ \mathbb{D}_{\mathcal{P}}(\mathcal{E})$ for any $\mathcal{E} \in F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$. The functor $\mathbb{D}_{Y, \mathcal{P}}$ only depends on the couple by using⁽⁵⁾ Lemma 1.2.7. This functor, called the *dual functor*, is denoted by

$$\mathbb{D}_{\mathbb{Y}}: F\text{-}D_{\text{ovhol}}^b(\mathbb{Y}/K) \rightarrow F\text{-}D_{\text{ovhol}}^b(\mathbb{Y}/K).$$

⁽⁴⁾ The “ c -” stands for “complete”.

⁽⁵⁾ We may check that this is not a vicious circle.

(ii) The tensor product $\mathbb{L}_{\mathcal{O}_{\mathcal{P},\mathbb{Q}}}^\dagger$ is defined on $F-D_{\text{ovhol}}^b(\mathcal{D}_{\mathcal{P},\mathbb{Q}}^\dagger)$. By [Car08], the bifunctor $\mathbb{L}_{\mathcal{O}_{\mathcal{P},\mathbb{Q}}}^\dagger[-\dim(\mathcal{P})]$ is independent of such a choice and is denoted by

$$\widetilde{\otimes}_{\mathbb{Y}}: F-D_{\text{ovhol}}^b(\mathbb{Y}/K) \times F-D_{\text{ovhol}}^b(\mathbb{Y}/K) \rightarrow F-D_{\text{ovhol}}^b(\mathbb{Y}/K).$$

This is called the *twisted tensor product functor*. For simplicity, we sometimes abbreviate $\widetilde{\otimes}_{\mathbb{Y}}$ by $\widetilde{\otimes}$.

(iii) We define the *tensor product* by

$$(-) \otimes_{\mathbb{Y}} (-) := \mathbb{D}_{\mathbb{Y}}(\mathbb{D}_{\mathbb{Y}}(-) \widetilde{\otimes}_{\mathbb{Y}} \mathbb{D}_{\mathbb{Y}}(-)): D_{\text{ovhol}}^b(\mathbb{Y}/K) \times D_{\text{ovhol}}^b(\mathbb{Y}/K) \rightarrow D_{\text{ovhol}}^b(\mathbb{Y}/K).$$

Remark. In the spirit of Riemann-Hilbert correspondence, our $-\widetilde{\otimes}-$ corresponds to $\mathbb{D}(\mathbb{D}(-) \otimes \mathbb{D}(-))$ in the theory of constructible sheaves, which justifies our notation.

1.1.7. Let $u: \mathbb{Y} \rightarrow \mathbb{Y}'$ be a morphism of couples. We have the following cohomological functors.

(i) We have *ordinary* and *extraordinary pull-back* by u :

$$u^+, u^!: F-D_{\text{ovhol}}^b(\mathbb{Y}'/K) \rightarrow F-D_{\text{ovhol}}^b(\mathbb{Y}/K).$$

They are constructed as follows: choose a morphism of l.p. frames $\widetilde{u} = (\star, \star, g, \star): (Y, X, \mathcal{P}, \mathcal{Q}) \rightarrow (Y', X', \mathcal{P}', \mathcal{Q}')$ of u . Then $\widetilde{u}^! := \mathbb{R}\Gamma_Y^\dagger \circ g^!$ does not depend on the choices of l.p. frames. We define $u^!$ to be $\widetilde{u}^!$. Using this, we define $\widetilde{u}^+ := \mathbb{D}_{Y,\mathcal{P}} \circ \widetilde{u}^! \circ \mathbb{D}_{Y',\mathcal{P}'}$, and $u^+ := \mathbb{D}_{\mathbb{Y}} \circ u^! \circ \mathbb{D}_{\mathbb{Y}'}$.

(ii) If u is complete (cf. Definition 1.1.2), we have the *ordinary* and *extraordinary direct image* by u :

$$u_!, u_+: F-D_{\text{ovhol}}^b(\mathbb{Y}/K) \rightarrow F-D_{\text{ovhol}}^b(\mathbb{Y}'/K).$$

There are constructed as follows: choose a morphism of l.p. frames $\widetilde{u} = (\star, \star, g, \star): (Y, X, \mathcal{P}, \mathcal{Q}) \rightarrow (Y', X', \mathcal{P}', \mathcal{Q}')$ for u . Then we put $\widetilde{u}_+ := g_+$, which does not depend on the choices and define $u_+ := \widetilde{u}_+$. As usual, we put $\widetilde{u}_! := \mathbb{D}_{Y',\mathcal{P}'} \circ u_+ \circ \mathbb{D}_{Y,\mathcal{P}}$, and $u_! := \mathbb{D}_{\mathbb{Y}'} \circ u_+ \circ \mathbb{D}_{\mathbb{Y}}$.

1.1.8. Let us define some extra functors.

(i) Let $\mathbb{Y} = (Y, X)$ and $\mathbb{Y}' = (Y', X')$ be couples. Let $\mathbb{Y}'' := (Y \times Y', X \times X')$, and denote by $p^{(\iota)}: \mathbb{Y}'' \rightarrow \mathbb{Y}^{(\iota)}$ the projections. We define the *exterior tensor product* by

$$(-) \boxtimes_K (-) := p^!(-) \widetilde{\otimes}_{\mathbb{Y}''} p^!(-): D_{\text{ovhol}}^b(\mathbb{Y}/K) \times D_{\text{ovhol}}^b(\mathbb{Y}'/K) \rightarrow D_{\text{ovhol}}^b(\mathbb{Y}''/K).$$

(ii) Let $\mathbb{Y} = (Y, X)$ be a couple. Let U be a subscheme of Y , \overline{U} be the closure of U in X , $\mathbb{U} := (U, \overline{U})$ and $u: \mathbb{U} \rightarrow \mathbb{Y}$ be the canonical c-immersion. We put

$$\mathbb{R}\Gamma_{\mathbb{U}}^\dagger := u_+ \circ u^!: F-D_{\text{ovhol}}^b(\mathbb{Y}/K) \rightarrow F-D_{\text{ovhol}}^b(\mathbb{Y}/K).$$

Moreover, we put $(^\dagger U) := \mathbb{R}\Gamma_{Y \setminus U}^\dagger$. If U_1 and U_2 are subschemes of Y , we have $\mathbb{R}\Gamma_{U_1}^\dagger \circ \mathbb{R}\Gamma_{U_2}^\dagger \xrightarrow{\sim} \mathbb{R}\Gamma_{U_1 \cap U_2}^\dagger$ and $(^\dagger U_1) \circ (^\dagger U_2) \xrightarrow{\sim} (^\dagger U_1 \cup U_2)$. When Z is closed subscheme of Y , we have the following *localization triangle* of functors:

$$\mathbb{R}\Gamma_Z^\dagger \rightarrow \text{id} \rightarrow (^\dagger Z) \xrightarrow{+1}.$$

(iii) Finally, let (Y, X, \mathcal{P}) be a frame, and $\mathcal{E} \in D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$. For a subscheme Y' of Y , let X' be the closure of Y' in X and $j = (\star, \star, \text{id}): (Y', X', \mathcal{P}) \rightarrow (Y, X, \mathcal{P})$ be the morphism of frames. We define $\mathcal{E}|_{Y'}$ to be $j^!(\mathcal{E})$.

1.1.9. We recall the following useful properties:

(i) Let $u: \mathbb{Y} \rightarrow \mathbb{Y}'$ be a morphism of couples. From [Car08, 2.1.9], for any $\mathcal{E}, \mathcal{F} \in F-D_{\text{ovhol}}^b(\mathbb{Y}'/K)$, we have the canonical isomorphism in $F-D_{\text{ovhol}}^b(\mathbb{Y}/K)$:

$$u^!(\mathcal{E} \widetilde{\otimes}_{\mathbb{Y}'} \mathcal{F}) \xrightarrow{\sim} u^!(\mathcal{E}) \widetilde{\otimes}_{\mathbb{Y}} u^!(\mathcal{F}). \quad (1.1.9.1)$$

(ii) Let $u: \mathbb{Y} \rightarrow \mathbb{Y}'$ be a c-proper morphism of couples. From the relative duality isomorphism together with [Abe13, 4.14], we get the isomorphism $u_! \xrightarrow{\sim} u_+$.

Lemma 1.1.10. *Let $u: \mathbb{Y} \rightarrow \mathbb{Y}'$ be a complete morphism of couples. We have adjoint pairs (u^+, u_+) and $(u_!, u^!)$.*

Proof. By transitivity with the composition, we reduce to the case where u is a c-open immersion and u is c-proper. In the first case, we may check easily that we have adjoint pairs (u^+, u_+) (and then $(u_!, u^!)$ by duality). In the second case, from the relative duality isomorphism, we need to check that we have adjoint pairs $(u_+, u^!)$, which follows from [Vir04]. \blacksquare

1.2 t-structures

Let $\mathbb{Y} = (Y, X)$ be a couple (cf. Definition 1.1.2). Choose an l.p. frame $(Y, X, \mathcal{P}, \mathcal{Q})$ of \mathbb{Y} . Let Z be a closed subvariety of P so that $Z := X \setminus Y$. We set $\mathcal{U} := \mathcal{P} \setminus Z$. Let us denote by $D^{\leq 0}(\mathcal{D}_{\mathcal{U}, \mathbb{Q}}^\dagger)$ (resp. $D^{\geq 0}(\mathcal{D}_{\mathcal{U}, \mathbb{Q}}^\dagger)$) the strictly full subcategory of $D^b(\mathcal{D}_{\mathcal{U}, \mathbb{Q}}^\dagger)$ consisting of complexes \mathcal{E} such that, for any $j \geq 1$ (resp. for any $j \leq -1$), we have $\mathcal{H}^j(\mathcal{E}) = 0$. We denote by $\tau_{\leq 0}: D^b(\mathcal{D}_{\mathcal{U}, \mathbb{Q}}^\dagger) \rightarrow D^{\leq 0}(\mathcal{D}_{\mathcal{U}, \mathbb{Q}}^\dagger)$ and $\tau_{\geq 0}: D^b(\mathcal{D}_{\mathcal{U}, \mathbb{Q}}^\dagger) \rightarrow D^{\geq 0}(\mathcal{D}_{\mathcal{U}, \mathbb{Q}}^\dagger)$ the usual truncation functors.

Definition 1.2.1. (i) Let $D^{\leq 0}(Y, \mathcal{P}/K)$ (resp. $D^{\geq 0}(Y, \mathcal{P}/K)$) be the strictly full subcategory of $D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ of complexes \mathcal{E} such that $\mathcal{E}|_{\mathcal{U}} \in D^{\leq 0}(\mathcal{D}_{\mathcal{U}, \mathbb{Q}}^\dagger)$ (resp. $\mathcal{E}|_{\mathcal{U}} \in D^{\geq 0}(\mathcal{D}_{\mathcal{U}, \mathbb{Q}}^\dagger)$).

(ii) Let $\tau_{\leq 0}^{(Y, \mathcal{P})}: D_{\text{ovhol}}^b(Y, \mathcal{P}/K) \rightarrow D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ and $\tau_{\geq 0}^{(Y, \mathcal{P})}: D_{\text{ovhol}}^b(Y, \mathcal{P}/K) \rightarrow D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ be the functors defined by $\tau_{\leq 0}^{(Y, \mathcal{P})} := (\dagger Z) \circ \tau_{\leq 0}$, $\tau_{\geq 0}^{(Y, \mathcal{P})} := (\dagger Z) \circ \tau_{\geq 0}$.

(iii) For any $n \in \mathbb{Z}$, we put $D^{\leq n}(Y, \mathcal{P}/K) := D^{\leq 0}(Y, \mathcal{P}/K)[-n]$, $D^{\geq n}(Y, \mathcal{P}/K) := D^{\geq 0}(Y, \mathcal{P}/K)[-n]$, $\tau_{\leq n}^{(Y, \mathcal{P})} := [-n] \circ \tau_{\leq 0}^{(Y, \mathcal{P})} \circ [n]$ and $\tau_{\geq n}^{(Y, \mathcal{P})} := [-n] \circ \tau_{\geq 0}^{(Y, \mathcal{P})} \circ [n]$.

Lemma 1.2.2. *Let $\phi: \mathcal{E} \rightarrow \mathcal{F}$ be a homomorphism of $D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$. Then, ϕ is an isomorphism if and only if $\phi|_{\mathcal{U}}$ is.*

Proof. A mapping cone \mathcal{G} of ϕ is an object of $D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ with support in Z . Thus, we have $\mathcal{G} \xleftarrow{\sim} \mathbb{R}\Gamma_Z^\dagger(\mathcal{G}) = 0$. \blacksquare

Lemma 1.2.3. *Let $\mathcal{E} \in D^{\leq n}(Y, \mathcal{P}/K)$ (resp. $\mathcal{E} \in D^{\geq n}(Y, \mathcal{P}/K)$). The canonical homomorphism $\tau_{\leq n}^{(Y, \mathcal{P})}(\mathcal{E}) \rightarrow \mathcal{E}$ (resp. $\mathcal{E} \rightarrow \tau_{\geq n}^{(Y, \mathcal{P})}(\mathcal{E})$) is an isomorphism. In particular, the essential image of $\tau_{\leq n}^{(Y, \mathcal{P})}$ (resp. $\tau_{\geq n}^{(Y, \mathcal{P})}$) is $D^{\leq n}(Y, \mathcal{P}/K)$ (resp. $D^{\geq n}(Y, \mathcal{P}/K)$).*

Proof. This follows from Lemma 1.2.2. \blacksquare

Proposition 1.2.4. *The functors $\tau_\star^{(Y, \mathcal{P})}$ define a t-structure, called the canonical t-structure, on $D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$:*

- (i) *For any homomorphism $f: \mathcal{E} \rightarrow \mathcal{F}$ such that $\mathcal{E} \in D^{\leq 0}(Y, \mathcal{P}/K)$, $\mathcal{F} \in D^{\geq 1}(Y, \mathcal{P}/K)$, we have $f = 0$.*
- (ii) *We have the inclusions $D^{\geq 1}(Y, \mathcal{P}/K) \subset D^{\geq 0}(Y, \mathcal{P}/K)$, $D^{\leq 0}(Y, \mathcal{P}/K) \subset D^{\leq 1}(Y, \mathcal{P}/K)$.*
- (iii) *For any $\mathcal{E} \in D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$, we have the distinguished triangle*

$$\tau_{\leq 0}^{(Y, \mathcal{P})}(\mathcal{E}) \rightarrow \mathcal{E} \rightarrow \tau_{\geq 1}^{(Y, \mathcal{P})}(\mathcal{E}) \xrightarrow{+1}.$$

Proof. Since $\tau_{\leq 0}^{(Y, \mathcal{P})}(\mathcal{F})$ has its support in Z , $\tau_{\leq 0}^{(Y, \mathcal{P})}(\mathcal{F}) = 0$. From Lemma 1.2.3, we get that f factorizes through $\tau_{\leq 0}^{(Y, \mathcal{P})}(f) = 0$. Hence we get the first assertion. The other ones are obvious. \blacksquare

Definition 1.2.5. We denote by $\text{Ovhol}(Y, \mathcal{P}/K)$ the heart of the canonical t-structure on $D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$. We define for any integer n , the n -th cohomology functor $\mathcal{H}_t^n: D_{\text{ovhol}}^b(Y, \mathcal{P}/K) \rightarrow \text{Ovhol}(Y, \mathcal{P}/K)$ by putting $\mathcal{H}_t^n(\mathcal{E}) := \tau_{\leq 0}^{(Y, \mathcal{P})} \tau_{\geq 0}^{(Y, \mathcal{P})}(\mathcal{E}[n])$ for any $\mathcal{E} \in D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$.

Remark 1.2.6. (i). Let $\mathcal{E} \in D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$. Then we get a natural morphism $\mathcal{H}^n(\mathcal{E}) \rightarrow \mathcal{H}_t^n(\mathcal{E})$, where $\mathcal{H}^n(\mathcal{E})$ is the usual n -th cohomology functor. This homomorphism is not an isomorphism. However, the induced morphism

$$(\dagger Z) \circ \mathcal{H}^n(\mathcal{E}) \rightarrow \mathcal{H}_t^n(\mathcal{E}), \quad (1.2.6.1)$$

is an isomorphism since this is the case outside Z .

(ii). For any $n \in \mathbb{Z}$, $\mathcal{E} \in D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$, we have $\mathcal{H}_t^n(\mathcal{E})|_{\mathcal{U}} \xrightarrow{\sim} \mathcal{H}^n(\mathcal{E}|_{\mathcal{U}})$. Hence, from Lemma 1.2.2, $\mathcal{H}_t^n(\mathcal{E}) = 0$ if and only if $\mathcal{H}^n(\mathcal{E}|_{\mathcal{U}}) = 0$.

(iii). By definition, the category $\text{Ovhol}(Y, \mathcal{P}/K)$ is the strictly full subcategory of $D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ consisting of complexes \mathcal{E} such that $\mathcal{H}^0(\mathcal{E}|_{\mathcal{U}}) \xrightarrow{\sim} \mathcal{E}|_{\mathcal{U}}$. By Lemma 1.2.3, the category $\text{Ovhol}(Y, \mathcal{P}/K)$ is also the strictly full subcategory of $D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ of complexes \mathcal{E} such that $\mathcal{H}_t^0(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$. By (1.2.6.1), this implies the inclusion $\text{Ovhol}(Y, \mathcal{P}/K) \subset D^{\geq 0}(X, \mathcal{P}/K)$. Indeed, by using Mayer-Vietoris exact triangles and an induction on the number of divisors whose intersection is Z , we reduce to the case where Z is a divisor.

Next, we complete [Car11b, 4.2.3] with the following lemma:

Lemma 1.2.7. Let $u = (\text{id}, \star, \star, \star): (Y, X', \mathcal{P}', \mathcal{Q}') \rightarrow (Y, X, \mathcal{P}, \mathcal{Q})$ be a complete morphism of l.p. frames.

1. For any $\mathcal{E} \in \text{Ovhol}(Y, \mathcal{P}/K)$, $\mathcal{E}' \in \text{Ovhol}(Y, \mathcal{P}'/K)$, for any $n \in \mathbb{Z} \setminus \{0\}$,

$$\mathcal{H}_t^n(u^!(\mathcal{E}')) = 0, \quad \mathcal{H}_t^n(u_+(\mathcal{E})) = 0.$$

2. The functors $\mathcal{H}_t^0 u^!$ and $\mathcal{H}_t^0 u_+$ (resp. $u^!$ and u_+) induce equivalence of categories between $\text{Ovhol}(Y, \mathcal{P}/K)$ and $\text{Ovhol}(Y, \mathcal{P}'/K)$ (resp. between $F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ and $F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}'/K)$).

3. We have canonical isomorphisms $u_! \xrightarrow{\sim} u_+$ and $u^! \xrightarrow{\sim} u^+$.

Proof. By Remark 1.2.6, the two first statements follows by [Car11b, 4.2.3]. Concerning the last one, we remark that the isomorphism $u_! \xrightarrow{\sim} u_+$ is equivalent to the other one $u^! \xrightarrow{\sim} u^+$. Moreover, by transitivity, we come down to the case where g is proper and is an open immersion. In the proper case we have $u_! \xrightarrow{\sim} u_+$, and in the open immersion case, we have $u^! \xrightarrow{\sim} u^+$. \blacksquare

Definition 1.2.8. (i) Let \mathbb{Y} be a couple. Take an l.p. frame $(Y, X, \mathcal{P}, \mathcal{Q})$ of \mathbb{Y} . By Lemma 1.2.7, the t-structure of $D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ is compatible with canonical equivalence of categories, and independent on the choices of \mathcal{P} and \mathcal{Q} (cf. Definition 1.1.5). This makes $D_{\text{ovhol}}^b(\mathbb{Y}/K)$ a triangulated category with t-structure. Its heart is denoted by $\text{Ovhol}(\mathbb{Y}/K)$. The objects in $\text{Ovhol}(\mathbb{Y}/K)$ are called *overholonomic arithmetic \mathcal{D} -modules* on \mathbb{Y} , or *overholonomic $\mathcal{D}_{\mathbb{Y}}^\dagger$ -modules*. For simplicity, we often call them $\mathcal{D}_{\mathbb{Y}}^\dagger$ -modules, or *modules* on \mathbb{Y} .

(ii) Let $u: \mathbb{Y}' \rightarrow \mathbb{Y}$ be a morphism of couples. We define two functors $\text{Ovhol}(\mathbb{Y}/K) \rightarrow \text{Ovhol}(\mathbb{Y}'/K)$ by $u^{!0} := \mathcal{H}_t^0 \circ u^!$ and $u^{+0} := \mathcal{H}_t^0 \circ u^+$. When u is complete, we define two functors $\text{Ovhol}(\mathbb{Y}'/K) \rightarrow \text{Ovhol}(\mathbb{Y}/K)$ by setting $u_+^0 := \mathcal{H}_t^0 \circ u_+$ and $u_!^0 := \mathcal{H}_t^0 \circ u_!$.

(iii) Let $j: (Y', X') \rightarrow (Y, X)$ be a morphism of couples such that $X' \rightarrow X$ is an immersion. We denote $j^{!0}$ by $|_{(Y', X')}$.

Remark 1.2.9. Let $j: (Y, X') \rightarrow (Y, X)$ be a morphism of couples such that $X' \rightarrow X$ is an open immersion. Then a sequence $\mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}''$ in $\text{Ovhol}((Y, X)/K)$ is t-exact if and only if so is the sequence $\mathcal{E}'|_{(Y, X')} \rightarrow \mathcal{E}|_{(Y, X')} \rightarrow \mathcal{E}''|_{(Y, X')}$. Moreover, an object $\mathcal{E} \in \text{Ovhol}(\mathbb{Y}/K)$ is 0 if and only if $\mathcal{E}|_{(Y, X')}$ is 0. In particular, the functor $|_{(Y, X')}: \text{Ovhol}((Y, X)/K) \rightarrow \text{Ovhol}((Y, X')/K)$ is faithful.

Definition 1.2.10. Let \mathbb{Y}, \mathbb{Y}' be two couples and $\phi: D_{\text{ovhol}}^b(\mathbb{Y}/K) \rightarrow D_{\text{ovhol}}^b(\mathbb{Y}'/K)$ be a functor. We say that ϕ is *left t-exact* (resp. *right t-exact*, resp. *t-exact*) if, for any $\mathcal{E} \in \text{Ovhol}(Y, \mathcal{P}/K)$ and any integer $n \in \mathbb{Z}$ such that $n \leq -1$ (resp. $n \geq 1$, resp. $n \neq 0$), we have $\mathcal{H}_t^n \circ \phi(\mathcal{E}') = 0$.

1.2.11. Let $(Y^{(')}, X^{(')}) \rightarrow (Y'', X'')$ be morphisms of couples. We denote $(Y, X) \times_{(Y'', X'')} (Y', X') := (Y \times_{Y''} Y', X \times_{X''} X')$. Let $\{\mathbb{Y}_i\}_{i \in I}$ be a c-open covering of \mathbb{Y} , namely, we have the c-open immersions $(Y_i, X_i) = \mathbb{Y}_i \rightarrow \mathbb{Y}$ such that $\{Y_i\}_{i \in I}$ is an open covering of Y . We put $\mathbb{Y}_{ij} := \mathbb{Y}_i \times_{\mathbb{Y}} \mathbb{Y}_j$ and $\mathbb{Y}_{ijk} := \mathbb{Y}_{ij} \times_{\mathbb{Y}} \mathbb{Y}_k$. For any $i, j, k \in I$, we denote by $u_i: \mathbb{Y}_i \rightarrow \mathbb{Y}$, $u_{ij}: \mathbb{Y}_{ij} \rightarrow \mathbb{Y}_i$, $v_{ij} := u_i \circ u_{ij}$, $u_{ijk}: \mathbb{Y}_{ijk} \rightarrow \mathbb{Y}_{ij}$ the induced c-open immersions.

Now, we define $\text{Ovhol}(\{\mathbb{Y}_i\}_{i \in I}/K)$ to be the category whose objects are the data of objects $\mathcal{E}_i \in \text{Ovhol}(\mathbb{Y}_i/K)$ endowed with isomorphisms of the form $\theta_{ji}: u_{ij}^{!0}(\mathcal{E}_i) \xrightarrow{\sim} u_{ji}^{!0}(\mathcal{E}_j)$ which satisfy the cocycle condition $u_{ijk}^{!0}(\theta_{ki}) = u_{jki}^{!0}(\theta_{kj}) \circ u_{ij}^{!0}(\theta_{ij})$, for any $i, j, k \in I$. A morphism $(\mathcal{E}_i, \theta_{ij}) \rightarrow (\mathcal{E}'_i, \theta'_{ij})$ in $\text{Ovhol}(\{\mathbb{Y}_i\}_{i \in I}/K)$ is a collection of morphisms $f_i: \mathcal{E}_i \rightarrow \mathcal{E}'_i$ in $\text{Ovhol}(\mathbb{Y}_i/K)$ such that $u_{ji}^{!0}(f_i) \circ \theta_{ij} = \theta'_{ij} \circ u_{ji}^{!0}(f_j)$, for any $i, j \in I$.

Lemma (Gluing). *The canonical functor $\text{Ovhol}(\mathbb{Y}/K) \rightarrow \text{Ovhol}(\{\mathbb{Y}_i\}_{i \in I}/K)$ is an equivalence of categories.*

Proof. By the quasi-compactness of Y , we may assume I to be a finite set. We can construct a canonical quasi-inverse functor as follows: let $\{\mathcal{E}_i, \theta_{ij}\} \in \text{Ovhol}(\{\mathbb{Y}_i\}_{i \in I}/K)$. We denote by $d_1, d_2: \prod_{i \in I} u_{i+}^0 \mathcal{E}_i \rightarrow \prod_{i, j \in I} v_{ij+}^0 u_{ij}^{!0}(\mathcal{E}_i)$ the morphisms such that d_1 is induced by $u_{i+}^0 \mathcal{E}_i \rightarrow v_{ij+}^0 u_{ij}^{!0}(\mathcal{E}_i)$ and d_2 is induced by $u_{j+}^0 \mathcal{E}_j \rightarrow v_{ji+}^0 u_{ji}^{!0}(\mathcal{E}_j) \xrightarrow{\sim} v_{ij+}^0 u_{ij}^{!0}(\mathcal{E}_i)$. Then the canonical quasi-inverse functor is by definition the kernel of $d_1 - d_2$. \blacksquare

1.2.12. Recall the situation of paragraph 1.1.4. Let \mathcal{T} be a triangulated category with t-structure τ_* . Assume given an additive endofunctor F^* of \mathcal{T} which is assumed to be t-exact. The truncation functor τ_* lifts to a functor from $F\text{-}\mathcal{T}$ to itself since F^* is assumed to be t-exact. By construction, τ_* commutes with ϱ . As usual, we put $\mathcal{H}^n := \tau_{\leq n} \circ \tau_{\geq n}$, and are able to define the “heart” of $F\text{-}\mathcal{T}$, in an obvious manner. This heart is nothing but $F\text{-Heart}(\mathcal{T})$, and in particular, it is abelian.

Now, the Frobenius pull-back functor on the category $D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ is t-exact. Hence, we may apply the abstract non-sense above, and get that the heart of $F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ (resp. $F\text{-}D_{\text{ovhol}}^b(\mathbb{Y}/K)$) is $F\text{-Ovhol}(Y, \mathcal{P}/K)$ (resp. $F\text{-Ovhol}(\mathbb{Y}/K)$). The category $F\text{-Ovhol}(\mathbb{Y}/K)$ is noetherian and artinian. Indeed, we reduce to the case where $Y = X = P$ in which case it follows by [Ber02, 5.4.3].

1.2.13. Let $\mathbb{Y} = (Y, X)$ be a couple such that Y is smooth. Let $Z := X \setminus Y$. When there exists a divisor W of P such that $Z = W \cap X$, a strictly full subcategory $\text{Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K)$ of the category of coherent $D_{\mathcal{P}}^{\dagger}(\dagger Z)_{\mathbb{Q}}$ -modules with support in X is defined in [Car11a]. Moreover, we have the equivalence

$$\text{sp}_+: \text{Isoc}^{\dagger}(Y, X/K) \xrightarrow{\sim} \text{Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K).$$

We may endow with Frobenius structure: $F\text{-Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K)$ is a full subcategory of $F\text{-Ovhol}(Y, \mathcal{P}/K)$ (cf. [CT12]), and the similar equivalence holds.

Let us generalize this equivalence to arbitrary Z . We define the category $F\text{-Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K)$ to be the strictly full subcategory of $F\text{-Ovhol}(Y, \mathcal{P}/K)$ consisting of objects \mathcal{E} such that $\mathcal{E}|_{\mathcal{U}} \in F\text{-Isoc}^{\dagger\dagger}(Y, \mathcal{U}/K)$. The category $\text{Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K)$ only depends on \mathbb{Y} and K and can be denoted by $\text{Isoc}^{\dagger\dagger}(\mathbb{Y}/K)$. Moreover, we denote by $D_{\text{isoc}}^b(Y, \mathcal{P}/K)$ the full subcategory of $D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ consisting of complexes \mathcal{E} such that $\mathcal{H}_t^j(\mathcal{E}) \in \text{Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K)$ for any integer j . The category $D_{\text{isoc}}^b(Y, \mathcal{P}/K)$ depends only on \mathbb{Y} and K , and can be denoted by $D_{\text{isoc}}^b(\mathbb{Y}/K)$.

(i) We have the equivalence of categories:

$$F\text{-Isoc}^{\dagger}(\mathbb{Y}/K) \cong F\text{-Isoc}^{\dagger\dagger}(\mathbb{Y}/K).$$

Indeed, by gluing lemma 1.2.11 and the gluing for overconvergent isocrystals, we can reduce to the case where Z is the intersection of a divisor on P with X , in which case we have already recalled above.

(ii) Let $E \in F\text{-Isoc}^{\dagger}(\mathbb{Y}/K)$, y be a closed point of Y . Let d be the dimension of Y locally around y . By using [Abe13, 5.6], we have the isomorphisms

$$i_y^+(\text{sp}_+(E)) \xrightarrow{\sim} i_y^!(\text{sp}_+(E))(d)[2d] \xrightarrow{\sim} \text{sp}_+(i_y^*(E))(d)[d], \quad (1.2.13.1)$$

where $i_y: (\{y\}, \{y\}) \rightarrow \mathbb{Y}$ denotes the canonical morphism.

1.3 Properties of six functors for couples

In this subsection, we prove some fundamental properties for six functors defined in the previous subsection.

Proposition 1.3.1. *Let \mathbb{Y} be a couple. Then the dual functor $\mathbb{D}_{\mathbb{Y}}$ is t -exact (cf. Definition 1.2.10).*

Proof. Let $(Y, X, \mathcal{P}, \mathcal{Q})$ be an l.p. frame of \mathbb{Y} . By Remark 1.2.6.(ii), we reduce to the case where $Y = X$ and then from Berthelot-Kashiwara theorem to the case where $Y = X = P$. The lemma follows from the exactness of $\mathbb{D}_{\mathcal{P}}$ (see [Vir00]). \blacksquare

Proposition 1.3.2. *Let $u: \mathbb{Y} = (Y, X) \rightarrow \mathbb{Y}' = (Y', X')$ be a morphism of couples.*

(i) If u is c -smooth of relative dimension d such that the fibers are equidimensional, then $u^+[d]$ and $u^![-d]$ are t -exact.

(ii) If u is a c -immersion, then $u^!$ (resp. u^+) is left t -exact (resp. right t -exact). Moreover, we have the canonical isomorphism $u^! \circ u_+ \xrightarrow{\sim} \text{id}$.

(iii) (Kashiwara's theorem) Suppose u is a c -closed immersion. Then u_+ is t -exact. Moreover, u_+ (resp. u_+^0) is fully faithful. The objects of the essential image of u_+ (resp. u_+^0) are called “with support in \mathbb{Y} ”. Restricted to objects with support in \mathbb{Y} , the functor $u^!$ is t -exact and $u^!$ (resp. $u^{!0}$) is canonically a quasi-inverse to u_+ (resp. u_+^0).

Proof. Let us show (i). By Remark 1.2.6.(ii), we may assume that u is of the form $u = (b, b): (Y, Y) \rightarrow (Y', Y')$. The problem is local both on Y and Y' , so we may assume that there exists a factorization $Y \rightarrow Y' \times \mathbb{A}^d \xrightarrow{p} Y'$ where the first morphism is étale and the second is the projection. For the projection case, take a closed embedding $Y' \hookrightarrow \mathcal{P}'$, and let $\tilde{p}: \mathcal{P}' \times \widehat{\mathbb{A}}^d \rightarrow \mathcal{P}'$. Then $p^!$ is induced by $\tilde{p}^!$, which is exact after shifting by the flatness of \tilde{p} . Thus, we may assume that b is étale. Take a closed point $y \in Y$. It suffices to show the exactness around y . By the structure theorem of étale morphism (cf. EGA IV, Theorem 18.4.6), locally around y and $b(y)$, we can take the following cartesian diagram

$$\begin{array}{ccc} Y & \hookrightarrow & \mathcal{P} \\ \downarrow b & \square & \downarrow \tilde{b} \\ Y' & \hookrightarrow & \mathcal{P}' \end{array}$$

where \mathcal{P} and \mathcal{P}' are smooth formal schemes, and \tilde{b} is étale. By the flatness of \tilde{b} , the exactness follows.

Let us show (ii) and (iii). By duality, we may concentrate on showing the $u^!$ case. By using Lemma 1.2.7, we may assume that u is of the form $(b, \text{id}): (Y, X) \rightarrow (Y', X)$, which reduce easily to already known cases (e.g. see Kashiwara's theorem proven by Berthelot in [Ber02]). \blacksquare

Proposition 1.3.3. *(i) Using the notation of paragraph 1.1.8 (i), we have a canonical isomorphism $(-)\boxtimes_K (-) \cong p^+(-) \otimes_{\mathbb{Y}''} p'^+(-)$.*

(ii) Exterior tensor products are t -exact.

Proof. Let us show (i). Take l.p. frames $(Y, X, \mathcal{P}, \mathcal{Q})$ and $(Y', X', \mathcal{P}', \mathcal{Q}')$. Let $\tilde{p}^{(\iota)}: \mathcal{P} \times \mathcal{P}' \rightarrow \mathcal{P}^{(\iota)}$. Let $\mathcal{E}^{(\iota)} \in D_{\text{ovhol}}^b(\mathcal{D}_{\mathcal{P}^{(\iota)}, \mathbb{Q}}^\dagger)$. Consider the homomorphism

$$\mathbb{R}\Gamma_{Y \times Y'}^\dagger(\mathcal{E} \boxtimes_K \mathcal{E}') \rightarrow \mathcal{E} \boxtimes_K \mathcal{E}'.$$

When $\mathcal{E}^{(\iota)} \in D_{\text{ovhol}}^b(Y^{(\iota)}, X/K)$, this is an isomorphism. Indeed, the right side is supported on $\overline{Y \times Y'}$, the closure of $Y \times Y'$ in $P \times P'$. Putting $Z := \overline{Y \times Y'} \setminus Y \times Y'$, it remains to show that $({}^\dagger Z)(\mathcal{E} \boxtimes \mathcal{E}') = 0$. This follows by (1.1.9.1). Thus the proposition is reduced to showing that there exists an isomorphism $\mathbb{D}_{\mathcal{P} \times \mathcal{P}'}(\mathcal{E} \boxtimes \mathcal{E}') \cong \mathbb{D}_{\mathcal{P}}(\mathcal{E}) \boxtimes \mathbb{D}_{\mathcal{P}'}(\mathcal{E}')$ for $\mathcal{E}^{(\iota)} \in D_{\text{perf}}(\mathcal{D}_{\mathcal{P}^{(\iota)}, \mathbb{Q}}^\dagger)$. Since $\widehat{\mathcal{D}}_{\mathcal{P}}^{(m)}$ is of finite cohomological dimension, by [Vir00, I.4], it suffices to show that $\mathbb{D}_{\mathcal{P} \times \mathcal{P}'}^{(m)}(\mathcal{E}^{(m)} \boxtimes \mathcal{E}'^{(m)}) \cong \mathbb{D}_{\mathcal{P}}^{(m)}(\mathcal{E}^{(m)}) \boxtimes \mathbb{D}_{\mathcal{P}'}^{(m)}(\mathcal{E}'^{(m)})$ for $\mathcal{E}^{(\iota(m))} \in D_{\text{perf}}(\widehat{\mathcal{D}}_{\mathcal{P}^{(\iota)}}^{(m)})$. By passing to the limit, we are reduced to the following fact, whose verification is easy: let

\mathcal{R} be a commutative ring on a topos, and \mathcal{A}, \mathcal{B} be flat \mathcal{R} -algebras such that \mathcal{R} is the center of them. Let $\mathcal{M} \in D_{\text{perf}}^b(\mathcal{A})$, $\mathcal{N} \in D_{\text{perf}}^b(\mathcal{B})$. Putting $\mathcal{C} := \mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}$, which is a \mathcal{R} -algebra, we have an isomorphism

$$\mathbb{R}\mathcal{H}\text{om}_{\mathcal{A}}(\mathcal{M}, \mathcal{A}) \otimes_{\mathcal{R}}^{\mathbb{L}} \mathbb{R}\mathcal{H}\text{om}_{\mathcal{B}}(\mathcal{N}, \mathcal{B}) \cong \mathbb{R}\mathcal{H}\text{om}_{\mathcal{C}}(\mathcal{M} \otimes_{\mathcal{R}}^{\mathbb{L}} \mathcal{N}, \mathcal{C}).$$

Compatibility of Frobenius is also standard.

Let us check (ii). For an immersion u of couples, u_+ and $\tilde{\otimes}$ commute by (1.1.9.1). Thus, by dévissage, the lemma is reduced to the overconvergent F -isocrystal case. In this case, the verification is easy. \blacksquare

1.3.4. Let $u = (\star, a): \mathbb{Y} = (Y, X) \rightarrow (Y', X') = \mathbb{Y}'$ be a complete morphism of couples. In this paragraph, we construct the canonical homomorphism $\theta_u: u_! \rightarrow u_+$.

Let $\mathcal{E} \in D_{\text{ovhol}}^b(\mathbb{Y}/K)$. The morphism u factors as $\mathbb{Y} \xrightarrow{\iota} (U, X) = \mathbb{U} \xrightarrow{u'} \mathbb{Y}'$, where $U := a^{-1}(Y')$. Let $Z := U \setminus Y$. Since $\mathbb{D}_{\mathbb{Y}} = (\dagger Z) \circ \mathbb{D}_{\mathbb{U}}$ and since Z is closed in Y , we get the canonical homomorphism $\mathbb{D}_{\mathbb{U}} \circ \mathbb{D}_{\mathbb{Y}}(\mathcal{E}) \rightarrow \mathbb{D}_{\mathbb{Y}} \circ \mathbb{D}_{\mathbb{Y}}(\mathcal{E}) \cong \mathcal{E}$, which yields the functorial homomorphism $\iota_!(\mathcal{E}) \rightarrow \iota_+(\mathcal{E})$. Moreover, since u' is c-proper, we have the relative duality isomorphism $u'_! \xrightarrow{\sim} u'_+$. We define

$$\theta_u: u_! \cong u'_! \circ \iota_! \rightarrow u'_! \circ \iota_+ \cong u'_+ \circ \iota_+ \cong u_+.$$

In order to check the transitivity at Proposition 1.3.7, we need the following lemmas.

Lemma 1.3.5. *Let (X, \mathcal{A}) be a ringed space (where \mathcal{A} is not necessary commutative), and $\mathcal{B} \rightarrow \mathcal{C}$ be a homomorphism in the derived category $D^b(\mathcal{A} \otimes_{\mathbb{Z}} \mathcal{A})$. For any $\mathcal{A} \otimes_{\mathbb{Z}} \mathcal{A}$ -complex \mathcal{E} , and $\star \in \{\mathcal{B}, \mathcal{C}\}$, we put $\mathbb{D}_{\star}(\mathcal{E}) := \mathbb{R}\mathcal{H}\text{om}_{\mathcal{A}}(\mathcal{E}, \star)$, where we used the first \mathcal{A} -module structure of \star to take $\mathcal{H}\text{om}$, and consider $\mathbb{D}_{\star}(\mathcal{E})$ as a left \mathcal{A} -module using the second \mathcal{A} -module structure of \star . Let $\beta: \mathcal{E} \rightarrow \mathbb{D}_{\mathcal{B}} \circ \mathbb{D}_{\mathcal{B}}(\mathcal{E})$ be the canonical homomorphism. Then the following diagram is commutative:*

$$\begin{array}{ccc} \mathbb{D}_{\mathcal{B}} \circ \mathbb{D}_{\mathcal{B}} \circ \mathbb{D}_{\mathcal{C}}(\mathcal{E}) & \xleftarrow{\quad} & \mathbb{D}_{\mathcal{B}} \circ \mathbb{D}_{\mathcal{B}} \circ \mathbb{D}_{\mathcal{B}}(\mathcal{E}) \xrightarrow{\mathbb{D}_{\mathcal{B}}(\beta)} \mathbb{D}_{\mathcal{B}}(\mathcal{E}) \\ \beta(\mathbb{D}_{\mathcal{C}}(\mathcal{E})) \uparrow & & \uparrow \beta(\mathbb{D}_{\mathcal{B}}(\mathcal{E})) \\ \mathbb{D}_{\mathcal{C}}(\mathcal{E}) & \xleftarrow{\quad} & \mathbb{D}_{\mathcal{B}}(\mathcal{E}). \end{array}$$

Proof. We only need to check the commutativity of the right triangle, whose verification is left to the reader. \blacksquare

Lemma 1.3.6. *Consider the following left commutative diagram of l.p. frames:*

$$\begin{array}{ccc} (Y, X, \mathcal{P}, \mathcal{Q}) & \xrightarrow{u=(b, \star, f, g)} & (Y''', X', \mathcal{P}', \mathcal{Q}') \\ \downarrow \iota=(i, \text{id}, \text{id}, \text{id}) & & \downarrow \iota'=(i', \text{id}, \text{id}, \text{id}) \\ (Y'', X, \mathcal{P}, \mathcal{Q}) & \xrightarrow{u'=(b', \star, f, g)} & (Y', X', \mathcal{P}', \mathcal{Q}'), \end{array} \quad \begin{array}{ccc} \iota'_! \circ u_! & \xrightarrow{\iota'_!(\theta_u)} & \iota'_! \circ u_+ \xrightarrow{\theta_{\iota'}(u_+)} \iota'_+ \circ u_+ \\ \sim \downarrow & & \downarrow \sim \\ u'_! \circ \iota_! & \xrightarrow{u'_!(\theta_{\iota})} & u'_! \circ \iota_+ \xrightarrow{\theta_{u'}(\iota_+)} u'_+ \circ \iota_+ \end{array}$$

Assume that all the morphisms in the diagram are complete, and b, b' are proper. Then the right diagram above of functors from $F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ to $F\text{-}D_{\text{ovhol}}^b(Y', \mathcal{P}'/K)$ is commutative.

Proof. Consider the following diagram, where we omit subscripts \mathcal{P} and \mathcal{P}' from \mathbb{D} :

$$\begin{array}{ccc} \mathbb{D} \circ \mathbb{D}_{Y'''} \circ \mathbb{D}_{Y'''} \circ f_+ \circ \mathbb{D}_Y & \xrightarrow{\quad} & \mathbb{D} \circ \mathbb{D}_{Y'''} \circ f_+ \\ \sim \downarrow & \nearrow & \downarrow \\ \mathbb{D} \circ f_+ \circ \mathbb{D}_Y & \xrightarrow{\quad} & \mathbb{D} \circ f_+ \circ \mathbb{D} \\ \sim \uparrow & \searrow & \downarrow \\ \mathbb{D} \circ f_+ \circ \mathbb{D}_{Y''} \circ \mathbb{D}_{Y''} \circ \mathbb{D}_Y & \xrightarrow{\quad} & \mathbb{D} \circ f_+ \circ \mathbb{D}_{Y''}. \end{array}$$

By taking the functor $\mathbb{R}\Gamma_{Y'}^\dagger$, we get the desired diagram, so it suffices to check the commutativity of the big diagram above. The commutativity of small diagrams except for the one with \star are easy. To check the commutativity of the triangle diagrams marked with \star , we need to check the commutativity of the following diagram:

$$\begin{array}{ccc} \mathbb{D}_{Y''} \circ \mathbb{D}_{Y''} \circ \mathbb{D}_Y & \longleftarrow & \mathbb{D}_{Y''} \\ \uparrow \sim & \swarrow & \\ \mathbb{D}_Y & & \end{array}$$

Take a set of divisors $\{T_i\}_{1 \leq i \leq r''}$ of P such that $X \setminus (\bigcap_{1 \leq i \leq r''} T_i) = Y^{(r'')}$ for some $r'' \leq r$. Consider the following Čech type complexes

$\mathcal{D}^{(r'')} =$

$$\mathbf{s} \left[\begin{array}{ccccccc} \mathcal{D}_{\mathcal{P}, \mathbb{Q}}^\dagger & \rightarrow & \bigoplus_{1 \leq k \leq r^{(r'')}} \mathcal{D}_{\mathcal{P}, \mathbb{Q}}^\dagger(\dagger T_k) & \rightarrow & \bigoplus_{1 \leq k_1 < k_2 \leq r^{(r'')}} \mathcal{D}_{\mathcal{P}, \mathbb{Q}}^\dagger(\dagger T_{k_1} \cup T_{k_2}) & \rightarrow \dots \rightarrow & \mathcal{D}_{\mathcal{P}, \mathbb{Q}}^\dagger(\dagger T_1 \cup \dots \cup T_{r^{(r'')}}) \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{D}_{\mathcal{P}, \mathbb{Q}}^\dagger & \longrightarrow & 0 & \longrightarrow & \dots & & \end{array} \right]$$

where $\mathcal{D}_{\mathcal{P}, \mathbb{Q}}^\dagger$ of the first line is placed at degree $(0, 0)$, and \mathbf{s} denotes the simple complex associated with the double complex. We have the canonical homomorphism of complexes $\mathcal{D} \rightarrow \mathcal{D}^{(r'')}$. Recall that

$$\mathbb{D}_{Y^{(r'')}}(-) \cong \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{\mathcal{P}, \mathbb{Q}}^\dagger}(-, \mathcal{D}^{(r'')}) \otimes_{\mathcal{O}_{\mathcal{P}}} \omega_{\mathcal{P}}^{-1}[d_P].$$

The commutativity follows from a lemma on general non-sense Lemma 1.3.5 above. \blacksquare

Proposition 1.3.7. *Let $u: \mathbb{Y} \rightarrow \mathbb{Y}'$, $u': \mathbb{Y}' \rightarrow \mathbb{Y}''$ be two complete morphisms of couples. We have a canonical isomorphism $\theta_{u'}(u_+) \circ u'_!(\theta_u) \cong \theta_{u' \circ u}$.*

Proof. This follows from Lemma 1.3.6. \blacksquare

Lemma 1.3.8. *Let $u: \mathbb{Y} \rightarrow \mathbb{Y}'$ be a complete morphism of couples. Then the following diagram of functors from $F\text{-}D_{\mathrm{ovhol}}^b(\mathbb{Y}/K)$ to $F\text{-}D_{\mathrm{ovhol}}^b(\mathbb{Y}'/K)$ is commutative.*

$$\begin{array}{ccc} u_! & \xrightarrow{\theta_u} & u_+ \\ \parallel & & \downarrow \sim \\ \mathbb{D}_{\mathbb{Y}'} u_+ \mathbb{D}_{\mathbb{Y}} & \xrightarrow{\mathbb{D}_{\mathbb{Y}'}(\theta_u)} & \mathbb{D}_{\mathbb{Y}'} u_! \mathbb{D}_{\mathbb{Y}}. \end{array}$$

Proof. Let $u = (\star, a)$, and consider the factorization $\mathbb{Y} = (Y, X) \xrightarrow{i} \mathbb{Y}'' := (a^{-1}(Y'), X) \xrightarrow{p} \mathbb{Y}' = (Y', X')$. Let $\mathcal{E} \in F\text{-}D_{\mathrm{ovhol}}^b(\mathbb{Y}/K)$. Consider the following diagram:

$$\begin{array}{ccccccc} u_! \mathcal{E} & \longrightarrow & p_+ i_! \mathcal{E} & \longrightarrow & p_+ \mathbb{D}_{\mathbb{Y}''} i_+ \mathbb{D}_{\mathbb{Y}} \mathcal{E} & \longrightarrow & \mathbb{D}_{\mathbb{Y}'} u_+ \mathbb{D}_{\mathbb{Y}} \mathcal{E} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ u_+ \mathcal{E} & \longrightarrow & p_+ i_+ \mathcal{E} & \longrightarrow & p_+ \mathbb{D}_{\mathbb{Y}''} i_! \mathbb{D}_{\mathbb{Y}} \mathcal{E} & \longrightarrow & \mathbb{D}_{\mathbb{Y}'} u_! \mathbb{D}_{\mathbb{Y}} \mathcal{E}. \end{array}$$

Since the outer two squares are commutative, it suffices to treat the $u = i$ case, and to show that the diagram

$$\begin{array}{ccccc} i_!(\mathcal{E}) & \xrightarrow{\theta_i(\mathcal{E})} & i_+(\mathcal{E}) & \xlongequal{\quad} & \mathcal{E} \\ \parallel & & \downarrow \sim & & \downarrow \sim \\ \mathbb{D}_{\mathbb{Y}''} i_+ \mathbb{D}_{\mathbb{Y}}(\mathcal{E}) & \xrightarrow{\mathbb{D}_{\mathbb{Y}''}(\theta_i(\mathbb{D}_{\mathbb{Y}}(\mathcal{E})))} & \mathbb{D}_{\mathbb{Y}''} i_! \mathbb{D}_{\mathbb{Y}}(\mathcal{E}) & \xlongequal{\quad} & \mathbb{D}_{\mathbb{Y}''} \mathbb{D}_{\mathbb{Y}''} \mathbb{D}_{\mathbb{Y}} \mathbb{D}_{\mathbb{Y}}(\mathcal{E}) \end{array} \quad (\star)$$

is commutative. We assume $u = i$, namely $X = X'$ and $Y \subset Y' \subset X$, in the following. Take an l.p. frame $(Y', X, \mathcal{P}, \mathcal{Q})$, and put $T := X \setminus Y$. We denote $\mathbb{D}_{Y(\cdot), \mathcal{P}}$ by $\mathbb{D}_{Y(\cdot)}$. The homomorphism α is the one induced by $\text{id} \rightarrow (\dagger T)$ and β is the biduality isomorphisms $\text{id} \xrightarrow{\sim} \mathbb{D}_Y \mathbb{D}_Y = (\dagger T) \mathbb{D}_{Y'} \mathbb{D}_Y$. Let us consider the following diagram of $F\text{-}\mathcal{D}_{\mathcal{P}, \mathcal{Q}}^\dagger$ -complexes:

$$\begin{array}{ccccccc}
\mathcal{E} & \xleftarrow{\theta(\mathcal{E})} & \mathbb{D}_{Y'} \mathbb{D}_Y(\mathcal{E}) & \xleftarrow{\mathbb{D}_{Y'}(\beta)} & \mathbb{D}_{Y'}(\dagger T) \mathbb{D}_{Y'} \mathbb{D}_Y \mathbb{D}_Y(\mathcal{E}) & \xrightarrow{\mathbb{D}_{Y'}(\alpha)} & \mathbb{D}_{Y'} \mathbb{D}_{Y'} \mathbb{D}_Y \mathbb{D}_Y(\mathcal{E}) \\
\parallel & & \downarrow \alpha & & \downarrow \alpha & & \sim \downarrow \alpha \\
\mathcal{E} & \xrightarrow[\beta]{\sim} & (\dagger T) \mathbb{D}_{Y'} \mathbb{D}_Y(\mathcal{E}) & \xleftarrow[\sim]{(\dagger T) \mathbb{D}_{Y'}(\beta)} & (\dagger T) \mathbb{D}_{Y'}(\dagger T) \mathbb{D}_{Y'} \mathbb{D}_Y \mathbb{D}_Y(\mathcal{E}) & \xrightarrow[\sim]{(\dagger T) \mathbb{D}_{Y'}(\alpha)} & (\dagger T) \mathbb{D}_{Y'} \mathbb{D}_{Y'} \mathbb{D}_Y \mathbb{D}_Y(\mathcal{E}).
\end{array} \quad (\star\star)$$

This diagram is commutative by definition and by functoriality. Composing the morphisms of the bottom and next the right vertical morphism (i.e. α^{-1}) of $(\star\star)$, we get an isomorphism $\gamma: \mathcal{E} \xrightarrow{\sim} \mathbb{D}_{Y'} \mathbb{D}_{Y'} \mathbb{D}_Y \mathbb{D}_Y(\mathcal{E})$. Since the diagram $(\star\star)$ is commutative, it is sufficient to check that this isomorphism γ is equal to the right arrow of (\star) . Let us consider the following diagram:

$$\begin{array}{ccc}
\mathbb{D}_{Y'} \mathbb{D}_Y(\mathcal{E}) & \xleftarrow[\sim]{\beta} & \mathcal{E} \\
\parallel & & \downarrow \sim \beta \circ \beta \\
\mathbb{D}_{Y'} \mathbb{D}_Y(\mathcal{E}) & \xleftarrow[\sim]{\mathbb{D}_{Y'}(\beta)} & \mathbb{D}_{Y'} \mathbb{D}_{Y'} \mathbb{D}_Y \mathbb{D}_Y(\mathcal{E})
\end{array} \quad (\star\star\star)$$

We remark that, in this case, the right vertical morphism of $(\star\star\star)$ is the right morphism of (\star) and that $\gamma = \mathbb{D}_{\mathcal{P}}(\beta) \circ \beta^{-1}$. Thus, it remains to check the commutativity of $(\star\star\star)$. For this, we argue as in the last part of the proof of Lemma 1.3.6. \blacksquare

Lemma 1.3.9. *Consider the following cartesian diagram D on the left of l.p. frames (i.e. the four underlying squares are cartesian):*

$$\begin{array}{ccc}
(Y''', X''', \mathcal{P}''', \mathcal{Q}''') & \xrightarrow{v'} & (Y, X, \mathcal{P}, \mathcal{Q}) \\
u' \downarrow & \square & \downarrow u \\
(Y'', X'', \mathcal{P}'', \mathcal{Q}'') & \xrightarrow{v} & (Y', X', \mathcal{P}', \mathcal{Q}'),
\end{array} \quad \begin{array}{ccccc}
\star & \xrightarrow{v'_2} & \star & \xrightarrow{v'_1} & \star \\
u'' \downarrow & \square & u' \downarrow & \square & \downarrow u \\
\star & \xrightarrow{v_2} & \star & \xrightarrow{v_1} & \star
\end{array}$$

such that u is complete (which implies the completeness of u' as well). Then we have a canonical base change isomorphism $b_D: v'_! u_+ \xrightarrow{\sim} u'_+ v_!^!$. This base change isomorphism is compatible with compositions with respect to v in the following sense: consider the diagram on the right above. The stars denote some l.p. frames, and u is assumed to be complete. Its right (resp. left) square is called D (resp. D') and the outer big diagram is called E . Then the composition

$$(v_1 \circ v_2)^! \circ u_+ \xrightarrow[\sim]{b_D} v_2^! \circ u'_+ \circ v_1^! \xrightarrow[\sim]{b_{D'}} u'_+ \circ (v_1^! \circ v_2^!)$$

is equal to b_E . Similarly the base change isomorphism is compatible with composition with respect to u as well.

Proof. Let $v: (Y, X, \mathcal{P}, \mathcal{Q}) \rightarrow (Y', X', \mathcal{P}', \mathcal{Q}')$ be a morphism of l.p. frames. The morphism v is said to be *cartesian* if $\mathcal{P} = \mathcal{Q} \times_{\mathcal{Q}'} \mathcal{P}'$, $X = Q \times_{Q'} X'$, $Y = Q \times_{Q'} Y'$, of *type I* if g is an immersion, and of *type II* if it is cartesian and g is smooth. We notice that v factors canonically as $(Y, X, \mathcal{P}, \mathcal{Q}) \xrightarrow{\iota} (Q \times Y', Q \times X', \mathcal{Q} \times \mathcal{P}', \mathcal{Q} \times \mathcal{Q}') \xrightarrow{v'} (Y', X', \mathcal{P}', \mathcal{Q}')$ where ι is of type I and v' is II. We can define the base change isomorphism for morphisms of type I and II individually. Indeed, for type I, the base change isomorphism is nothing but the transitivity of the local cohomology functor and for type II, this is [Abe13, 10.7]. These isomorphisms are compatible with the composition of u . The base change isomorphism for v is defined to be the composition.

It remains to show that the isomorphism is compatible with composition with respect to v . To construct such a natural transform, we may reduce to the following commutative and cartesian diagrams

$$\begin{array}{ccc} \star & \xrightarrow{v} & \star \\ & \searrow \iota' & \downarrow \iota \\ & & \star, \end{array} \quad \begin{array}{ccc} \star & \xrightarrow{\iota} & \star \\ & \searrow v' & \downarrow v \\ & & \star, \end{array} \quad \begin{array}{ccc} \star & \xrightarrow{\iota'} & \star \\ v' \downarrow & \square & \downarrow v \\ \star & \xrightarrow{\iota} & \star, \end{array}$$

where ι and ι' are morphisms of type I and v and v' are of type II, and construct a natural isomorphism between the composition of the base change isomorphism of ι , v' and ι' , v . The first one easily follows using the equivalence in Lemma 1.2.7, and the last one is similar to this. Let us check the second one. Let $v': (Y', X', \mathcal{P}', \mathcal{Q}') \rightarrow (Y, X, \mathcal{P}, \mathcal{Q})$ and $v: (Y'', X'', \mathcal{P}'', \mathcal{Q}'') \rightarrow (Y, X, \mathcal{P}, \mathcal{Q})$. Let i be the immersion $\mathcal{Q}' \hookrightarrow \mathcal{Q}''$ defining ι , and $\tilde{v}: \mathcal{Q}' \rightarrow \mathcal{Q}$ be the one defining v . Since v and v' are assumed to be cartesian, ι is cartesian as well, and ι' is nothing but i' . Thus the base change theorem is reduced to showing the following claim, whose verification is left to the reader.

Claim. *Keeping the notation, consider the following cartesian diagram of smooth formal schemes:*

$$\begin{array}{ccccc} \mathcal{P}' & \xrightarrow{i'} & \mathcal{P}'' & \xrightarrow{\tilde{v}'} & \mathcal{P} \\ f' \downarrow & \square & \downarrow & \square & \downarrow f \\ \mathcal{Q}' & \xrightarrow{i} & \mathcal{Q}'' & \xrightarrow{\tilde{v}} & \mathcal{Q}. \end{array}$$

Then the two canonical base change isomorphisms between functors

$$(\tilde{v} \circ i)^! \circ f_+ \cong f'_+ \circ (\tilde{v}' \circ i')^!: D_{\text{coh}}^b(\mathcal{D}_{\mathcal{P}, \mathbb{Q}}^\dagger) \rightarrow D^b(\mathcal{D}_{\mathcal{Q}, \mathbb{Q}}^\dagger)$$

are identical. ■

Proposition 1.3.10 (Base change). *Consider the following diagram of couples which induces an isomorphism $Y''' \cong Y \times_{Y'} Y''$:*

$$\begin{array}{ccc} Y''' & \xrightarrow{v'} & Y \\ u' \downarrow & \square & \downarrow u \\ Y'' & \xrightarrow{v} & Y', \end{array}$$

where u and u' are complete. Then we have a canonical isomorphism $v^! \circ u_+ \cong u'_+ \circ v'^!$. This isomorphism is compatible with composition with respect to both u and v .

Proof. Let $Y^* = (Y^*, X^*)$ where $*$ $\in \{\emptyset, I, II, III\}$. The diagram can be supplemented as follows:

$$\begin{array}{ccccc} (Y''', X''') & \xrightarrow{\alpha} & (Y''', X'' \times_{X'} X) & \xrightarrow{v''} & (Y, X) \\ & & u'' \downarrow & \square & \downarrow u \\ & & (Y'', X'') & \xrightarrow{v} & (Y', X'). \end{array}$$

Let us construct the base change isomorphism for cartesian diagram of couples: $v^! \circ u_+ \cong u'_+ \circ v'^!$. To construct this isomorphism, take a following cartesian diagram

$$\begin{array}{ccc} (Y''', X'' \times_{X'} X, \mathcal{P} \times_{\mathcal{P}'} \mathcal{P}'', \mathcal{Q} \times_{\mathcal{Q}'} \mathcal{Q}'') & \xrightarrow{\tilde{v}'} & (Y, X, \mathcal{P}, \mathcal{Q}) \\ \tilde{u}'' \downarrow & \square & \downarrow \tilde{u} \\ (Y'', X'', \mathcal{P}'', \mathcal{Q}'') & \xrightarrow{\tilde{v}} & (Y', X', \mathcal{P}', \mathcal{Q}') \end{array}$$

where \tilde{u} and \tilde{v} are morphisms of frames of u and v respectively such that the *morphism of formal schemes are smooth*. Then applying Lemma 1.3.9, we have a base change isomorphism $\tilde{v}^! \circ \tilde{u}_+ \cong \tilde{u}_+'' \circ \tilde{v}'''$. We need to check that the isomorphism does not depend on the choice of \tilde{u} and \tilde{v} . This follows easily from the compatibility of the composition.

Now, we define the desired base change isomorphism to be $v^! \circ u_+ \cong u_+'' \circ v''' \cong u_+'' \circ (\alpha_+ \circ \alpha^!) \circ v''' \cong u_+'' \circ v^!$. To check the compatibility, we need diagram chasing using Lemma 1.3.9, which is tedious but not difficult. ■

Lemma 1.3.11. *Let \mathbb{Y} be a couple, and $\mathcal{E} \in F-D_{\text{ovhol}}^b(\mathbb{Y}/K)$. Assume, for any closed point y of Y , we have $i_y^!(\mathcal{E}) = 0$, where $i_y: (\{y\}, \{y\}) \rightarrow \mathbb{Y}$ is the canonical morphism. Then $\mathcal{E} = 0$.*

Proof. From Lemma 1.2.2, one can suppose \mathbb{Y} is of the form (Y, Y) . We proceed by induction on the dimension of the support of \mathcal{E} . We may assume that the support of \mathcal{E} is Y . There exists a closed subscheme $Z \subset Y$ such that $U := Y \setminus Z$ is dense and smooth, and $j^! \mathcal{E}$ is in $F-D_{\text{isoc}}^b(U, Y/K)$ where $j: (U, Y) \rightarrow (Y, Y)$ (cf. paragraph 1.2.13). Since for $u \in U$ we have $i_u^! j^! \mathcal{E} = 0$ where $i_u: (\{u\}, \{u\}) \rightarrow (U, Y)$, we have $j^! \mathcal{E} = 0$. Consider the localization triangle $\mathbb{R}\Gamma_Z^\dagger(\mathcal{E}) \rightarrow \mathcal{E} \rightarrow (\dagger Z)(\mathcal{E}) \xrightarrow{+}$. Since for any $y \in Z$, $i_y^!(\mathbb{R}\Gamma_Z^\dagger(\mathcal{E})) = 0$, we have $\mathbb{R}\Gamma_Z^\dagger(\mathcal{E}) = 0$ by induction hypothesis. We conclude since $(\dagger Z)(\mathcal{E}) = j_+ j^! \mathcal{E} = 0$. ■

Proposition 1.3.12. *Let $u: \mathbb{Y}' \rightarrow \mathbb{Y}$ be a morphism of couples. If u is a c -universal homeomorphism (i.e. by EGA IV, Proposition 8.11.6, finite, surjective and radicial), then the functors u_+ and $u^!$ induce canonical equivalence of categories between $F-D_{\text{ovhol}}^b(\mathbb{Y}/K)$ and $F-D_{\text{ovhol}}^b(\mathbb{Y}'/K)$.*

Proof. Let $\mathcal{E} \in F-D_{\text{ovhol}}^b(\mathbb{Y}/K)$ and $\mathcal{E}' \in F-D_{\text{ovhol}}^b(\mathbb{Y}'/K)$. It is sufficient to check that adjunction morphisms $u_+ \circ u^!(\mathcal{E}) \rightarrow \mathcal{E}$ and $\mathcal{E}' \rightarrow u^! \circ u_+(\mathcal{E}')$ are isomorphisms. Let y be a closed point Y and $y' := u^{-1}(y)$. We denote by $i_y: y \hookrightarrow Y$ (resp. $i_{y'}: y' \hookrightarrow Y'$) the canonical closed immersion, and by $u_y: y' \rightarrow y$ the induced morphism, which is in fact an isomorphism since u is surjective, radicial, and k is perfect. From the base change theorem 1.3.10, we get $i_y^! \circ u_+ \xrightarrow{\sim} u_{y+} \circ i_{y'}^!$. By applying $i_y^!$ to the canonical morphism $u_+ \circ u^!(\mathcal{E}) \rightarrow \mathcal{E}$ we get the adjunction morphism $u_{y+} \circ u_y^!(i_y^! \mathcal{E}) \rightarrow i_y^! \mathcal{E}$, which is an isomorphism. From Lemma 1.3.11, this implies that the morphism $u_+ \circ u^!(\mathcal{E}) \rightarrow \mathcal{E}$ is an isomorphism. We may check similarly that $\mathcal{E}' \rightarrow u^! \circ u_+(\mathcal{E}')$ is an isomorphism as well. ■

Proposition 1.3.13. *Let $u: \mathbb{Y} \rightarrow \mathbb{Y}'$ be a complete morphism of couples.*

(i) *If u is c -affine, then u_+ (resp. $u_!$) is right t -exact (resp. left t -exact).*

(ii) *If u is c -quasi-finite, then u_+ (resp. $u_!$) is left t -exact (resp. right t -exact).*

Proof. Let us show (i). Take a morphism of l.p. frames $(\star, \star, \star, g): (Y, X, \mathcal{P}, \mathcal{Q}) \rightarrow (Y', X', \mathcal{P}', \mathcal{Q}')$ of u . From Remark 1.2.6, we may assume $Y' = P'$. Since the claim is local, we may assume \mathcal{P}' to be affine. Since Y is affine, we can take a closed immersion $Y \hookrightarrow \hat{\mathbb{A}}_{\mathcal{P}'}^n$ for some n . Since g is assumed to be smooth, the canonical morphism $\hat{\mathbb{P}}_{\mathcal{P}'}^n \times_{\mathcal{P}'} \mathcal{P} \rightarrow \mathcal{P}'$ is proper smooth, and we may consider the following commutative diagram of frames:

$$\begin{array}{ccccc} (Y, \bar{Y}, \hat{\mathbb{P}}_{\mathcal{P}'}^n) & \longleftarrow & (Y, \bar{Y}, \hat{\mathbb{P}}_{\mathcal{P}'}^n \times_{\mathcal{P}'} \mathcal{P}) & \longrightarrow & (Y, X, \mathcal{P}) \\ & \searrow u' & \downarrow & \swarrow & \\ & & (P', P', \mathcal{P}') & & \end{array}$$

Thus, we are reduced to showing the right exactness of u'_+ . Let $D \subset \hat{\mathbb{P}}_{\mathcal{P}'}^n$ be the divisor at infinity. By construction, $Y \hookrightarrow \hat{\mathbb{P}}_{\mathcal{P}'}^n \setminus D = \hat{\mathbb{A}}_{\mathcal{P}'}^n$ is a closed immersion, and we may assume that $Y = \mathbb{A}_{\mathcal{P}'}^n$. Then the proposition follows by [NH97, 5.4.1].

Let us show (ii). By exactness of the dual functor, it suffices to show the u_+ case. First, consider the case where u is a c -open immersion. By Lemma 1.2.7, we can suppose $X = X'$. Then, the claim follows from Remark 1.2.6.(iii). The case where u is a c -closed immersion is treated in Proposition 1.3.2. Finally, let us show the general case by the induction on the dimension of X . We may assume that the image of b is dense in Y' . By using the localization triangle, we may shrink Y , and assume that Y is integral. By Proposition 1.3.12 and using Zariski main theorem, we may assume that there exists an open dense subscheme $U' \subset Y'$

such that $u^{-1}(U') \rightarrow U$ is finite étale. Using the proven immersion case, we may shrink Y' , and reduced to showing the case where Y' is smooth and u finite étale. In this case, we see easily that u_+ is t-exact. ■

1.3.14. Let Y be a realizable variety. Take a couple (Y, X) such that X is proper. Then we define the category $(F-)D_{\text{ovhol}}^b(Y/K) := (F-)D_{\text{ovhol}}^b(Y, X/K)$. This does not depend on the choice of such a couple up to canonical equivalence of categories with t-structure. Indeed, take another couple (Y, X') with X' is proper. Let X'' be the closure of Y in $X \times X'$. We obtain two morphisms $(Y, X'') \rightarrow (Y, X^{(\prime)})$. Then, using Lemma 1.2.7 the independence follows.

With these categories, we get six functors formalism on the category of realizable varieties. Let $f: Y \rightarrow Y'$ be a morphism of realizable varieties. We have the following properties:

- (i). Dual functor on Y , denoted by \mathbb{D}_Y , is defined thanks to Lemma 1.2.7.3.
- (ii). Tensor functor on Y , denoted by $\tilde{\otimes}_Y$ or simply $\tilde{\otimes}$.
- (iii). Push-forward for f , denoted by f_+ is defined by the transitivity of the push-forward for couples.
- (iv). Extraordinary pull-back for f , denoted by $f^!$ is defined by the transitivity as well.
- (v). Extraordinary push-forward and ordinary pull-back for f , denoted by $f_!$ and f^+ , is defined by (i), (iii), (iv).
- (vi). We have a canonical homomorphism $f_! \rightarrow f_+$ and this is an isomorphism if f is proper (see paragraph 1.3.4).
- (vii). We have a base change isomorphism by Proposition 1.3.10.
- (viii). We have adjoint pairs (f^+, f_+) and $(f_!, f^!)$ by Lemma 1.1.10.

1.4 Intermediate extensions

In this subsection, unless otherwise stated, we let $u = (b, a): \mathbb{Y} = (Y, X) \rightarrow \mathbb{Y}' = (Y', X')$ be a c-immersion of couples.

Definition 1.4.1. For $\mathcal{E} \in F\text{-Ovhol}(\mathbb{Y}/K)$, we have the homomorphism $\theta_{u, \mathcal{E}}^0 := \mathcal{H}_t^0(\theta_{u, \mathcal{E}}): u_!^0 \mathcal{E} \rightarrow u_+^0 \mathcal{E}$ (see Definition 1.2.8 and paragraph 1.3.4 for the notation). We define

$$u_{!+}(\mathcal{E}) := \text{Im}(\theta_{u, \mathcal{E}}^0: u_!^0 \mathcal{E} \rightarrow u_+^0 \mathcal{E}).$$

This defines a functor $u_{!+}: F\text{-Ovhol}(\mathbb{Y}/K) \rightarrow F\text{-Ovhol}(\mathbb{Y}'/K)$, and it is called the *intermediate extension functor*.

Remark 1.4.2. (i) It follows from Proposition 1.3.13 that the functor $u_{!+}$ preserves injections and surjections.

(ii) Since the functor $\mathbb{D}_{\mathbb{Y}'}$ is t-exact on the category $F\text{-Ovhol}(\mathbb{Y}'/K)$ by Proposition 1.3.1, for any $\mathcal{E} \in F\text{-Ovhol}(\mathbb{Y}/K)$, we get the canonical isomorphisms:

$$\begin{aligned} \mathbb{D}_{\mathbb{Y}'}(\ker(\theta_{u, \mathcal{E}}^0)) &\xrightarrow{\sim} \text{coker}(\mathbb{D}_{\mathbb{Y}'}(\theta_{u, \mathcal{E}}^0)), & \mathbb{D}_{\mathbb{Y}'}(\text{coker}(\theta_{u, \mathcal{E}}^0)) &\xrightarrow{\sim} \ker(\mathbb{D}_{\mathbb{Y}'}(\theta_{\mathcal{E}}^0)), \\ \mathbb{D}_{\mathbb{Y}'}(\text{Im}(\theta_{u, \mathcal{E}}^0)) &\xrightarrow{\sim} \text{Im}(\mathbb{D}_{\mathbb{Y}'}(\theta_{u, \mathcal{E}}^0)). \end{aligned}$$

Moreover, when u is c-affine, both functors $u_+, u_!: F\text{-Ovhol}(\mathbb{Y}/K) \rightarrow F\text{-Ovhol}(\mathbb{Y}'/K)$ are t-exact by Proposition 1.3.13, and we do not need to take \mathcal{H}_t^0 in the definition of $u_{!+}$.

Corollary 1.4.3. Let $\mathcal{E} \in F\text{-Ovhol}(\mathbb{Y}/K)$. We have the canonical isomorphisms:

$$u_{!+}(\mathcal{E}) \xrightarrow{\sim} \mathbb{D}_{\mathbb{Y}'} \circ u_{!+} \circ \mathbb{D}_{\mathbb{Y}}(\mathcal{E}), \quad \ker(\theta_{u, \mathcal{E}}^0) \xrightarrow{\sim} \mathbb{D}_{\mathbb{Y}'}(\text{coker}(\theta_{u, \mathbb{D}_{\mathbb{Y}}(\mathcal{E})}^0)), \quad \text{coker}(\theta_{u, \mathcal{E}}^0) \xrightarrow{\sim} \mathbb{D}_{\mathbb{Y}'}(\ker(\theta_{u, \mathbb{D}_{\mathbb{Y}}(\mathcal{E})}^0)).$$

Proof. This is a consequence of Remark 1.4.2 (i) and Lemma 1.3.8. \blacksquare

Remark 1.4.4. Let $\mathcal{E} \in F\text{-Ovhol}(\mathbb{Y}/K)$. Let Y'' be the closure of Y in Y' and $Z := Y'' \setminus Y$. Since $\theta_{u,\mathcal{E}}^0$ is an isomorphism outside Z , this implies that the F -modules $\ker(\theta_{u,\mathcal{E}}^0)$ and $\text{coker}(\theta_{u,\mathcal{E}}^0)$ have their support in Z (cf. Proposition 1.3.2). Since u_+ and $\mathbb{R}\Gamma_Z^\dagger$ are left t-exact by Propositions 1.3.2 and 1.3.13, we get $0 = \mathcal{H}_t^0(\mathbb{R}\Gamma_Z^\dagger \circ u_+(\mathcal{E})) \cong \mathbb{R}^0\Gamma_Z^\dagger(u_+^0\mathcal{E})$. Now the exact sequence $0 \rightarrow u_{!+}(\mathcal{E}) \rightarrow u_+^0(\mathcal{E}) \rightarrow \text{coker}(\theta_{u,\mathcal{E}}^0) \rightarrow 0$ yields another short exact sequence:

$$0 \rightarrow \text{coker}(\theta_{u,\mathcal{E}}) \rightarrow \mathbb{R}^1\Gamma_Z^\dagger(u_{!+}(\mathcal{E})) \rightarrow \mathbb{R}^1\Gamma_Z^\dagger(u_+^0(\mathcal{E})) \rightarrow 0.$$

Suppose u is affine (e.g. when Z is a divisor of Y''). In this case, u_+ is t-exact and $\mathbb{R}\Gamma_Z^\dagger(u_+(\mathcal{E})) = 0$. Thus, we get the isomorphism:

$$\text{coker}(\theta_{u,\mathcal{E}}) \xrightarrow{\sim} \mathbb{R}\Gamma_Z^\dagger(u_{!+}(\mathcal{E}))[1]. \quad (1.4.4.1)$$

Lemma 1.4.5. Let $\mathbb{Y} \xrightarrow{u} \mathbb{Y}' \xrightarrow{u'} \mathbb{Y}''$ be c -immersions of couples.

(i) We have canonical equivalences of functors $u_+^0 \circ u_+^0 \cong (u' \circ u)_+^0$, and $u_!^0 \circ u_!^0 \cong (u' \circ u)_!^0$. This induces an equivalence $u_{!+}^0 \circ u_{!+}^0 \cong (u' \circ u)_{!+}^0$.

(ii) We have canonical isomorphisms $u^! \circ (u' \circ u)_+^0 \xrightarrow{\sim} u_+^0$, $u^! \circ (u' \circ u)_!^0 \xrightarrow{\sim} u_!^0$, and $u^! \circ (u' \circ u)_{!+}^0 \xrightarrow{\sim} u_{!+}^0$.

Proof. The first two isomorphisms follow by Proposition 1.3.13. Let $\mathcal{E} \in F\text{-Ovhol}(\mathbb{Y}/K)$. Using this, we have the canonical surjections: $u_!^0 \circ u_!^0(\mathcal{E}) \twoheadrightarrow u_!^0 \circ u_{!+}^0(\mathcal{E}) \twoheadrightarrow u_{!+}^0 \circ u_{!+}^0(\mathcal{E})$. We also have the canonical inclusions: $u_{!+}^0 \circ u_{!+}^0(\mathcal{E}) \hookrightarrow u_+^0 \circ u_{!+}^0(\mathcal{E}) \hookrightarrow u_+^0 \circ u_+^0(\mathcal{E})$. By functoriality, the composition of these homomorphisms is the canonical morphism $u_!^0 \circ u_!^0(\mathcal{E}) \rightarrow u_+^0 \circ u_+^0(\mathcal{E})$. Using the base change 1.3.10, the second part of the lemma is obvious. \blacksquare

Next proposition is a generalization of [Abe11, 2.1], and is generalized further at Theorem 1.4.9 below:

Lemma 1.4.6. Let $u: \mathbb{Y} \rightarrow \mathbb{Y}'$ be a c -open immersion. Let \mathcal{E}' be an irreducible object of $F\text{-Ovhol}(\mathbb{Y}'/K)$. Then $u^! \mathcal{E}'$ is either irreducible in $F\text{-Ovhol}(\mathbb{Y}/K)$ or 0.

Proof. Let $\alpha: u^! \mathcal{E}' \rightarrow \mathcal{F}$ be a homomorphism with non-zero kernel. By adjointness 1.1.10 and $u^! = u^+$ is t-exact, we get $\alpha': \mathcal{E}' \rightarrow u_+^0 \mathcal{F}$. Since $u^!(\ker(\alpha')) \cong \ker(\alpha) \neq 0$, we have $\ker(\alpha') \neq 0$. Since \mathcal{E}' is assumed to be irreducible, we have $\alpha' = 0$, and thus $\alpha = u^! \mathcal{E}' = 0$. Thus $u^! \mathcal{E}'$ is irreducible or 0. \blacksquare

Proposition 1.4.7. Let \mathcal{E} be an irreducible object of $F\text{-Ovhol}(\mathbb{Y}/K)$. Then:

(i) In the category $F\text{-Ovhol}(\mathbb{Y}'/K)$, $u_{!+}(\mathcal{E})$ is the unique irreducible subobject of $u_+^0(\mathcal{E})$.

(ii) In the category $F\text{-Ovhol}(\mathbb{Y}'/K)$, $u_{!+}(\mathcal{E})$ is the unique irreducible quotient of $u_!^0(\mathcal{E})$.

(iii) Let $j: \mathbb{Y}'' \rightarrow \mathbb{Y}'$ be a c -open immersion. Then, in $F\text{-Ovhol}(\mathbb{Y}''/K)$, either $j^!(u_+^0(\mathcal{E})) = 0$ or $j^!(u_{!+}(\mathcal{E}))$ is the unique irreducible subobject of $j^!(u_+^0(\mathcal{E}))$.

Proof. The proof is essentially the same as that of [BGK⁺87, VII.10.5]. Let us show (i). By using Berthelot-Kashiwara theorem 1.3.2.(iii) and Lemma 1.4.5, we may assume that u is a c -open immersion. We first claim that for any non-zero F -submodule \mathcal{F} of $u_+^0 \mathcal{E}$, $u^! \mathcal{F}$ is non-zero. Indeed, assume $u^! \mathcal{F} = 0$, which is equivalent to assuming $\mathbb{R}\Gamma_Y^\dagger(\mathcal{F}) = 0$ by Proposition 1.3.2.(ii). This means that \mathcal{F} has its support in $Z := \overline{Y} \setminus Y$ and then $\mathbb{R}^0\Gamma_Z^\dagger(\mathcal{F}) \xrightarrow{\sim} \mathcal{F}$. This implies that $\mathcal{F} \subset \mathbb{R}^0\Gamma_Z^\dagger(u_+^0(\mathcal{E}))$. By Remark 1.4.4, we have $\mathbb{R}^0\Gamma_Z^\dagger(u_+^0(\mathcal{E})) = 0$, thus $\mathcal{F} = 0$, which contradicts with the assumption.

Let us return to the proof. Now, let \mathcal{F} be an irreducible F -submodule of $u_+^0 \mathcal{E}$. By the left t-exactness of $u^!$, we have $0 \neq u^! \mathcal{F} \subset u^! u_+^0(\mathcal{E}) \cong \mathcal{E}$. Since \mathcal{E} is assumed to be irreducible, we have $u^! \mathcal{F} = \mathcal{E}$. Let \mathcal{F}' be another irreducible F -submodule of $u_+^0 \mathcal{E}$ (resp. $\mathcal{F}' := u_{!+}(\mathcal{E})$). Assume $\mathcal{F} \cap \mathcal{F}' = 0$. Left t-exactness of $u^!$ implies that $u^! \mathcal{F} \cap u^! \mathcal{F}' = 0$ in \mathcal{E} , which is impossible. Thus $\mathcal{F} = \mathcal{F}'$ (resp. $\mathcal{F} \subset u_{!+}(\mathcal{E})$), and \mathcal{F} is the unique irreducible F -submodule of $u_+^0 \mathcal{E}$. Let $\mathcal{Q} := u_{!+}(\mathcal{E})/\mathcal{F}$ be the quotient, and assume that this is not zero. This is supported in Z . We have

$$\mathbb{D}_{\mathbb{Y}'} \mathcal{Q} \subset \mathbb{D}_{\mathbb{Y}'}(u_{!+}(\mathcal{E})) \cong u_{!+}(\mathbb{D}_{\mathbb{Y}} \mathcal{E}) \subset u_+^0(\mathbb{D}_{\mathbb{Y}} \mathcal{E}).$$

The claim above shows that $u^!(\mathbb{D}_{\mathbb{Y}} \mathcal{Q})$ is non-zero, which is a contradiction with the fact that \mathcal{Q} is supported in Z . Thus $\mathcal{Q} = 0$, and \mathcal{F} is equal to $u_{!+}\mathcal{E}$, which completes the proof of (i). By duality, (ii) follows.

To show (iii), use Lemma 1.4.6, and base change of couples of functors $(j^!, u_+)$ and $(j^! \cong j^+, u_{!+})$. For details, we refer to [BGK⁺87]. \blacksquare

We can extend Proposition 1.4.7 as follows:

Proposition 1.4.8. *Let $\mathcal{E} \in F\text{-Ovhol}(\mathbb{Y}/K)$ and \mathcal{E}' be a subobject of $u_+(\mathcal{E})$ in $F\text{-Ovhol}(\mathbb{Y}'/K)$. Assume that the inclusion $u^!(\mathcal{E}') \hookrightarrow u^! \circ u_+(\mathcal{E}) = \mathcal{E}$ is an isomorphism. Then we have the canonical factorization $u_{!+}(\mathcal{E}) \subset \mathcal{E}' \subset u_+(\mathcal{E})$ of the inclusion $u_{!+}(\mathcal{E}) \subset u_+(\mathcal{E})$.*

Proof. The proof is analogue to that of Proposition 1.4.7.(i). \blacksquare

Theorem 1.4.9. (i) *Let \mathcal{E} be an irreducible object of $F\text{-Ovhol}(Y, X/K)$. Then there exists an open dense smooth subscheme Y' of Y and $\mathcal{E}' \in F\text{-Isoc}^{\dagger\dagger}(Y', X/K)$ such that $\mathcal{E} \cong u_{!+}(\mathcal{E}')$ where $u: (Y', X) \rightarrow (Y, X)$ is the canonical c-inclusion.*

(ii) *Let \mathbb{Y} be a couple, and $\{\mathbb{Y}_i\}_{i \in I}$ be an c-open covering of \mathbb{Y} (cf. paragraph 1.2.11). Let $u_i: \mathbb{Y}_i \rightarrow \mathbb{Y}$ be the c-open immersions. Then \mathcal{E} is semi-simple (resp. irreducible or 0) if and only if $u_i^! \mathcal{E}$ is semi-simple (resp. irreducible or 0) for any i .*

Proof. Copy the proof of [BGK⁺87, VII.10.6]. \blacksquare

1.5 Local theory on formal disk

In this subsection, we consider situation (A) in Notation and convention, and moreover, we assume k contains the field with q elements. We remark that this condition is satisfied in situation (B).

We denote by K^{ur} the maximal unramified extension of K . We put $\mathcal{K} := k((x))$, and let $G_{\mathcal{K}}$ be the absolute Galois group of $k((x))$. We often denote $G_{\mathcal{K}}$ by G , and the inertia group of G by I . For an integer n and a φ - K^{ur} -vector space V , $V(n)$ denotes the n -th Tate twist of V (cf. [Abe13, 2.7]).

Definition 1.5.1 ([Mar08, 3.1.3]). A *Deligne module* is a finite dimensional φ - K^{ur} -vector space (V, φ) endowed with semi-linear $G_{\mathcal{K}}$ -action σ commuting with the Frobenius action such that $\sigma|_I$ factors through a finite index subgroup of I , and a linear homomorphism of φ - K^{ur} -vector spaces $N: V(1) \rightarrow V$ which is nilpotent. A homomorphism between Deligne modules is a homomorphism compatible with other data in the obvious way. We denote such a Deligne module by (V, σ, φ, N) , and the category of Deligne modules is denoted by Del .

For a Deligne module $D := (V, \sigma, \varphi, N)$, we sometimes denote σ (resp. φ, N) by σ_D (resp. φ_D, N_D). For an integer n , we define the n -th Tate twist $D(n)$ of D by $(V, \sigma, q^{-n} \cdot \varphi, N)$.

1.5.2. Let (V, σ, φ, N) be a Deligne module. We will construct a canonical decomposition. The action of I on K^{ur} is trivial by definition, and in particular, the action of I on V is linear. We denote by $V^{I=1}$ the subset of V fixed by the action of I . Since the action of I on V is linear, $V^{I=1}$ is in fact a K^{ur} -vector space. Now, by the definition of Deligne module, there exists a subgroup $I' \subset I$ of finite index such that the action of I' on V is trivial, and thus, this induces the linear action of I/I' on V . For any $x \in V$, $\sum_{\sigma \in I/I'} \sigma(x)$ is in $V^{I=1}$. Thus, we get a K^{ur} -linear homomorphism

$$\pi := (\#I/I')^{-1} \sum_{\sigma \in I/I'} \sigma: V \rightarrow V^{I=1}.$$

We see easily that this does not depend on the choice of I' . For any $\tau \in G$, we have $\tau \cdot I = I \cdot \tau$. Thus we get, for any $x \in V$, $\tau((\sum_{\sigma \in I/I'} \sigma)(x)) = (\sum_{\sigma \in I/I'} \sigma)(\tau(x))$, which implies that $V^{I=1}$ possess a semi-linear action of G , which is nothing but the restriction of the G -action on V to $V^{I=1}$, and π commutes with G -actions. Since φ and N commute with G -action on V , these induce Frobenius and nilpotent operator on $V^{I=1}$, which

is also denoted by φ and N . Summing up, the data $(V^{I=1}, \sigma, \varphi, N)$ form a Deligne module, and π defines a homomorphism of Deligne modules.

By definition, the canonical inclusion $V^{I=1} \hookrightarrow V$ induces a homomorphism of Deligne modules, and π is a section of this inclusion. Thus, we obtain a canonical decomposition of Deligne modules

$$V \xrightarrow[(\pi, c)]{\sim} V^{I=1} \oplus V/V^{I=1},$$

where $c: V \rightarrow V/V^{I=1}$ is the projection. We denote $V/V^{I=1}$ by $V^{I \neq 1}$, and consider this as a submodule of V .

Definition 1.5.3 ([Cre12, 6.1]). A *solution data* is a set $(\Psi, \Phi, c, v, \{v(\sigma)\}_{\sigma \in I})$, where Ψ, Φ are Deligne modules, $c: \Psi \rightarrow \Phi$ (*canonical homomorphism*), $v: \Phi(1) \rightarrow \Psi$ (*variation homomorphism*), and $v(\sigma): \Phi \rightarrow \Psi$ (*Galois variation homomorphism*) are homomorphisms of Deligne modules such that

1. $N_\Psi = v \circ c$ and $N_\Phi = c \circ v$;
2. $\sigma_\Psi = 1 + v(\sigma) \circ c =$ and $\sigma_\Phi = 1 + c \circ v(\sigma)$.

Homomorphisms between solution data are defined in the obvious way. We denote the category, in fact an abelian category, of solution data by Sol .

Let V be a Deligne module such that σ_V is trivial for any $\sigma \in I$. Then $(V, 0, 0, 0, \{0\})$ defines a solution data. This solution data is denoted by $i_+(V)$.

1.5.4. One of the most important results of [Cre06] is that there exists an equivalence between the category of holonomic F - \mathcal{D}^{an} -modules and that of solution data. Let $\mathcal{S} := \text{Spf}(R[[x]])$, a formal disk. Crew defined the *ring of analytic differential operators* $\mathcal{D}_{\mathcal{S}, \mathbb{Q}}^{\text{an}}$ and that with poles $\mathcal{D}_{\mathcal{S}, \mathbb{Q}}^{\text{an}}(0)$. For simplicity, we denote these rings by \mathcal{D}^{an} and $\mathcal{D}^{\text{an}}(0)$. He proved fundamental properties of these rings. For the details, see *ibid.*. Let \mathcal{M} be a holonomic F - \mathcal{D}^{an} -module. He defined⁽⁶⁾ two functors Ψ and Φ by

$$\Psi(\mathcal{M}) := \text{Hom}_{\mathcal{D}^{\text{an}}}(\mathbb{D}(\mathcal{M}), \mathcal{B}), \quad \Phi(\mathcal{M}) := \text{Hom}_{\mathcal{D}^{\text{an}}}(\mathbb{D}(\mathcal{M}), \mathcal{C}),$$

where \mathcal{B} is the ring of hyperfunctions and \mathcal{C} is the microfunction space (cf. [ibid., 1.4, 6.1]). These are naturally Deligne modules, and the canonical homomorphism $c: \mathcal{B} \rightarrow \mathcal{C}$ and variation homomorphism $v: \mathcal{C} \rightarrow \mathcal{B}$ define canonical and variation homomorphisms between $\Psi(\mathcal{M})$ and $\Phi(\mathcal{M})$, which gives us a solution data $(\Phi(\mathcal{M}), \Psi(\mathcal{M}), \dots)$, and denoted by $\text{Da}(\mathcal{M})$. For the precise construction, see [ibid., §6].

Now, let us denote by $\text{Hol}(\mathcal{D}_{\mathcal{S}, \mathbb{Q}}^{\text{an}}(0))$ the category of holonomic F - $\mathcal{D}_{\mathcal{S}, \mathbb{Q}}^{\text{an}}(0)$ -modules, which is a subcategory of the category of holonomic F - $\mathcal{D}_{\mathcal{S}, \mathbb{Q}}^{\text{an}}$ -modules denoted by $\text{Hol}(\mathcal{D}_{\mathcal{S}, \mathbb{Q}}^{\text{an}})$. A direct consequence of local monodromy theorem is that the functor

$$\Psi: \text{Hol}(\mathcal{D}_{\mathcal{S}, \mathbb{Q}}^{\text{an}}(0)) \rightarrow \text{Del}$$

induces an equivalence of categories. Now, one of the main theorems of the Crew's paper is the following:

Theorem 1.5.5 ([Cre12, 7.1.1]). *The functor*

$$\text{Da}: \text{Hol}(\mathcal{D}_{\mathcal{S}, \mathbb{Q}}^{\text{an}}) \rightarrow \text{Sol}$$

induces an equivalence of categories.

A goal of this subsection is to describe $j_!\mathcal{M}$ and $j_+\mathcal{M}$ in terms of the solution data for a holonomic F - $\mathcal{D}^{\text{an}}(0)$ -module \mathcal{M} .

⁽⁶⁾Actually, he defined functors \mathbb{V} and \mathbb{W} . We modify the definition slightly after [AM11].

1.5.6. Let $\Psi := (\Psi, \sigma, \varphi, N)$ be a Deligne module. Then the data $(\Psi, \Psi, \text{id}, N, \{\sigma - 1\})$ defines a solution data, and we denote this by $j_!(\Psi)$.

We are able to construct another solution data out of Ψ . We put $\Psi' := \Psi^{I=1}(-1) \oplus \Psi^{I \neq 1}$. Let

$$\begin{aligned} c: \Psi &\cong \Psi^{I=1} \oplus \Psi^{I \neq 1} \xrightarrow{N \oplus \text{id}} \Psi^{I=1}(-1) \oplus \Psi^{I \neq 1} \cong \Psi', \\ v: \Psi'(1) &\cong \Psi^{I=1} \oplus \Psi^{I \neq 1}(1) \xrightarrow{\text{id} \oplus N} \Psi^{I=1} \oplus \Psi^{I \neq 1} \cong \Psi \\ v(\sigma): \Psi' &\cong \Psi^{I=1}(-1) \oplus \Psi^{I \neq 1} \xrightarrow{0 \oplus (\sigma - 1)} \Psi^{I=1} \oplus \Psi^{I \neq 1} \cong \Psi. \end{aligned}$$

Then we can check that $(\Psi, \Psi', c, v, \{v(\sigma)\})$ defines a solution data. We denote this data by $j_+(\Psi)$.

Let $\Psi^{I=1, N=0} := \text{Ker}(N: \Psi^{I=1} \rightarrow \Psi^{I=1}(-1))$, which is a Deligne module. We have the following exact sequence of solution data:

$$0 \rightarrow i_+(\Psi^{I=1, N=0}) \rightarrow j_!(\Psi) \rightarrow j_+(\Psi) \rightarrow i_+(\Psi^{I=1}(-1)/N\Psi^{I=1}) \rightarrow 0.$$

Proposition 1.5.7. Let $D := (\tilde{\Psi}, \tilde{\Phi}, c, v, \{v(\sigma)\})$ be a solution data. There exists a canonical isomorphism

$$\text{adj}: \text{Hom}_{\text{Del}}(\tilde{\Psi}, \Psi) \xrightarrow{\sim} \text{Hom}_{\text{Sol}}(D, j_+(\Psi)).$$

Proof. Let $c' := c|_{\tilde{\Psi}^{I \neq 1}}: \tilde{\Psi}^{I \neq 1} \rightarrow \tilde{\Phi}^{I \neq 1}$. Let us show that c' is an isomorphism. First, let us see the injectivity. Assume it were not injective, and let $x \in \tilde{\Psi}^{I \neq 1}$ be a non-zero element in $\text{Ker}(c')$. Then by assumption, there exists $\sigma \in I$ such that $\sigma(x) \neq x$. On the other hand, $\sigma(x) = (1 + v(\sigma) \circ c')(x) = x$, which is a contradiction. Now, to show the claim, we construct the inverse v' of c' . Let I' be the finite index subgroup of I which acts trivially on $\tilde{\Phi}^{I \neq 1}$. We put

$$v' := (\#I/I')^{-1} \sum_{\sigma \in I/I'} v(\sigma): \tilde{\Phi}^{I \neq 1} \rightarrow \tilde{\Psi}^{I \neq 1}.$$

Then, we see that

$$c' \circ v'(x) = (\#I/I')^{-1} \left(\sum_{\sigma \in I/I'} x - \sigma(x) \right) = x$$

where the last equality holds since there are no trivial subrepresentation of I in $\tilde{\Phi}^{I \neq 1}$. Since c' is an injection, v' is the inverse of c' . Thus the claim follows.

With these preparations, let us construct the homomorphism adj . Assume given a homomorphism $f: \tilde{\Psi} \rightarrow \Psi$. We put

$$v_1: \tilde{\Phi}^{I=1} \xrightarrow{v} \tilde{\Psi}^{I=1}(-1) \xrightarrow{f} \Psi^{I=1}(-1), \quad v_2: \tilde{\Phi}^{I \neq 1} \xrightarrow{v'} \tilde{\Psi}^{I \neq 1} \xrightarrow{f} \Psi^{I \neq 1}.$$

These homomorphisms define a homomorphism

$$f': \tilde{\Phi} \cong \tilde{\Phi}^{I=1} \oplus \tilde{\Phi}^{I \neq 1} \xrightarrow{v_1 \oplus v_2} \Psi^{I=1}(-1) \oplus \Psi^{I \neq 1} = \Psi'.$$

This defines the following homomorphism of solution data:

$$\begin{array}{ccc} D & & \tilde{\Psi} \begin{array}{c} \xrightarrow{\quad} \tilde{\Phi} \\ \xleftarrow{\quad} \end{array} \\ \text{adj}(f) \downarrow & & f \downarrow \begin{array}{c} \xrightarrow{\quad} \Psi' \\ \xleftarrow{\quad} \end{array} \\ j_+(\Psi) & & \Psi \begin{array}{c} \xrightarrow{\quad} \Psi' \\ \xleftarrow{\quad} \end{array} \end{array}$$

The verification of the compatibilities between homomorphisms is straightforward. As a result, we have a homomorphism

$$\text{adj}: \text{Hom}(\tilde{\Psi}, \Psi) \rightarrow \text{Hom}(D, j_+(\Psi)).$$

It is easy to check that this defines an isomorphism, and the details are left to the reader. ■

1.5.8. Let \mathcal{M} be a finite free differential \mathcal{R} -module with Frobenius structure. Consider the following complex:

$$0 \rightarrow \mathcal{M} \xrightarrow{\nabla} \mathcal{M} \otimes \Omega_{\mathcal{R}}^1 \rightarrow 0$$

where \mathcal{M} is placed at degree 0. We denote the i -th cohomology group by $H_{\text{loc}}^i(\mathcal{M})$. A result of Christol-Mebkhout [CM01] shows that these cohomology groups are finite dimensional vector space over K with the same dimension for $i = 0, 1$. Moreover, the canonical pairing

$$H_{\text{loc}}^0(\mathcal{M}) \otimes H_{\text{loc}}^1(\mathcal{M}^\vee) \rightarrow H_{\text{loc}}^1(\mathcal{R}) \cong K(-1)$$

is perfect from [Cre98, §5]. Finally let us remark that our $H_{\text{loc}}^0(M \otimes A(x))$ is denoted by $H^0(M \otimes A(x))$ by Crew in [Cre98, 10.7].

Theorem 1.5.9. *Let \mathcal{M} be a holonomic $F\text{-}\mathcal{D}^{\text{an}}(0)$ -module.*

(i) *We have $\Psi := \Psi(\mathcal{M}) \cong (\mathcal{B} \otimes_{\mathcal{R}} \mathcal{M})^{\partial=0}(1)$.*

(ii) *We have*

$$\text{Da}(j_!(\mathcal{M})) \cong j_!(\Psi(\mathcal{M})), \quad \text{Da}(j_+(\mathcal{M})) \cong j_+(\Psi(\mathcal{M})).$$

(iii) *We have the following isomorphisms of φ - K^{ur} -vector spaces with semi-linear $\text{Gal}(\bar{k}/k)$ -action:*

$$\begin{aligned} H_{\text{loc}}^0(\mathcal{M})(1) \otimes K^{\text{ur}} &\cong H^{-1}(i^+j_+(\mathcal{M})) \otimes K^{\text{ur}} \cong \Psi^{I=1, N=0}, \\ H_{\text{loc}}^1(\mathcal{M})(1) \otimes K^{\text{ur}} &\cong H^0(i^+j_+(\mathcal{M})) \otimes K^{\text{ur}} \cong \Psi^{I=1}(-1)/N\Psi^{I=1}. \end{aligned}$$

Proof. Let us show (i). We have

$$\Psi := \text{Hom}_{\mathcal{D}^{\text{an}}}(\mathbb{D}(\mathcal{M}), \mathcal{B}) \cong \text{Hom}_{\mathcal{D}^{\text{an}}(0)}(\mathbb{D}(\mathcal{M}), \mathcal{B}).$$

By using [Abe13, 3.12], we see that $\mathbb{D}(\mathcal{M}) \cong \mathcal{M}^\vee(-1)$. Thus, we get

$$\text{Hom}_{\mathcal{D}^{\text{an}}(0)}(\mathbb{D}(\mathcal{M}), \mathcal{B}) \cong \text{Hom}_{\mathcal{D}^{\text{an}}(0)}(\mathcal{M}^\vee, \mathcal{B})(1) \cong \text{Hom}_{\mathcal{D}^{\text{an}}(0)}(\mathcal{R}, \mathcal{M} \otimes \mathcal{B})(1),$$

and the first claim follows.

By [Cre12, 6.1.2], $\text{Da}(j_!(\mathcal{M})) \cong j_!(\Psi)$. Moreover, by Proposition 1.5.7, $\text{Da}(j_+(\mathcal{M}))$ is canonically isomorphic to $j_+(\Psi)$, and (ii) follows. Now, we have

$$\text{Hom}(\mathbb{D}(j_+\mathcal{M}), \mathcal{O}_{K^{\text{ur}}}^{\text{an}}) \cong H_{\text{loc}}^0(\mathcal{M})(1), \quad \text{Ext}^1(\mathbb{D}(j_+\mathcal{M}), \mathcal{O}_{K^{\text{ur}}}^{\text{an}}) \cong H_{\text{loc}}^1(\mathcal{M})(1),$$

by [AM11, 3.1.10]. We have the following diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Hom}(\mathbb{D}(\mathcal{M}), \mathcal{O}_{K^{\text{ur}}}^{\text{an}}) & \longrightarrow & \Psi(j_+\mathcal{M}) & \longrightarrow & \Phi(j_+\mathcal{M}) & \longrightarrow & \text{Ext}^1(\mathbb{D}(\mathcal{M}), \mathcal{O}_{K^{\text{ur}}}^{\text{an}}) \longrightarrow 0 \\ & & & & \sim \downarrow & & \downarrow \sim & & \\ 0 & \longrightarrow & \Psi^{I=1, N=0} & \longrightarrow & \Psi & \xrightarrow{c} & \Psi' & \longrightarrow & \Psi^{I=1}(-1)/N\Psi^{I=1} \longrightarrow 0, \end{array}$$

where the first row is exact by [Cre12, 6.1.1]. These show that

$$H_{\text{loc}}^0(\mathcal{M})(1) \otimes K^{\text{ur}} \cong \Psi^{I=1, N=0}, \quad H_{\text{loc}}^1(\mathcal{M})(1) \otimes K^{\text{ur}} \cong \Psi^{I=1}(-1)/N\Psi^{I=1}.$$

Finally, considering the following exact sequence, we conclude the proof.

$$0 \rightarrow i_+H^{-1}(i^+j_+(\mathcal{M})) \rightarrow j_!(\mathcal{M}) \rightarrow j_+(\mathcal{M}) \rightarrow i_+H^0(i^+j_+(\mathcal{M})) \rightarrow 0.$$

■

2 Mixed complexes

In this section, we define “mixed F -complexes”, and prove some basic properties. We consider situation (B) in Notation and convention.

2.1 Mixedness for F -isocrystals

2.1.1. In this subsection, let (Y, X, \mathcal{P}) be a frame such that Y is smooth. Let y be a closed point of Y with residue field $k(y)$. By abuse of notation, we denote by y the frame $(\mathrm{Spec}(k(y)), \mathrm{Spec}(k(y)), \mathcal{P})$, and by i_y the canonical morphism of frames $y \rightarrow (Y, X, \mathcal{P})$. We recall the notation of paragraph 1.1.8 (iii).

We start by recalling the following definition of ι -weight by Deligne [Del80]:

Definition 2.1.2. (i) Let $w \in \mathbb{R}$. A number α in $\overline{\mathbb{Q}}_p$ is said to be of ι -weight w if

$$|\iota(\alpha)| = q^{w/2}.$$

(ii) Let $\varphi\text{-Vect}_K$ be the category of pairs (V, φ) where V is a finite dimensional K -vector space and φ is an automorphism of V . Let $(a, a, f): (\mathrm{Spec}(k'), \mathrm{Spec}(k'), \mathcal{P}') \rightarrow (\mathrm{Spec}(k), \mathrm{Spec}(k), \mathrm{Spf} \mathcal{V})$ be a morphism of frames such that a is finite and f is the structural morphism of \mathcal{P}' . We have the canonical functor $f_+: F\text{-Ovhol}(\mathrm{Spec}(k'), \mathcal{P}'/K) \rightarrow \varphi\text{-Vect}_K$. Let $\mathcal{E}' \in F\text{-}D_{\mathrm{ovhol}}^b(\mathrm{Spec}(k'), \mathcal{P}'/K)$. We say that \mathcal{E}' is ι -pure if there exists a real number w , the *weight* of \mathcal{E}' , such that any eigenvalue α of the automorphism on $f_+(\mathcal{H}^j(\mathcal{E}'))$ is of ι -weight equal to $w + j$, for any integer j .

This definition is independant on the choice of \mathcal{P}' , i.e., if $\rho = (\mathrm{id}, \mathrm{id}, \star): (\mathrm{Spec}(k'), \mathrm{Spec}(k'), \mathcal{P}'') \rightarrow (\mathrm{Spec}(k'), \mathrm{Spec}(k'), \mathcal{P}')$ is a morphism of frames, then $\rho^!$ and ρ_+ induces quasi-inverse equivalences of categories between ι -pure of weight w objects of $F\text{-}D_{\mathrm{ovhol}}^b(\mathrm{Spec}(k'), \mathcal{P}'/K)$ and ι -pure of weight w objects of $F\text{-}D_{\mathrm{ovhol}}^b(\mathrm{Spec}(k'), \mathcal{P}''/K)$.

Definition 2.1.3. Let $\mathcal{E} \in F\text{-Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K)$.

1. We say that \mathcal{E} is ι -pure if there exists a real number w , called the *weight* of \mathcal{E} , such that for any closed point x of Y the pull-back $i_x^+ \mathcal{E}$ is ι -pure of weight w .
2. We say that \mathcal{E} is ι -mixed if all the constituents (i.e. irreducible subquotients) of \mathcal{E} are ι -pure.
3. We say that \mathcal{E} is ι -mixed of weight $\geq w$ (resp. $\leq w$) if \mathcal{E} is ι -mixed and if the weights of all the constituents of \mathcal{E} are $\geq w$ (resp. $\leq w$).

Remark 2.1.4. Let $E \in F\text{-Isoc}^{\dagger}(Y, X/K)$ be an overconvergent F -isocrystal and $\mathcal{E} := \mathrm{sp}_{X \hookrightarrow \mathcal{P}, +}(E)$ be the corresponding object of $F\text{-Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K)$. We assume that Y is pure of dimension d_Y . The F -isocrystal \mathcal{E} is ι -pure of weight⁽⁷⁾ w if and only if E is ι -pure of weight $w + d_Y$ in the sense of Crew or Kedlaya (see [Ked06, 5.1]) by (1.2.13.1).

Remark 2.1.5. Let $\mathcal{E} \in F\text{-Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K)$ and let \mathcal{E}' be a constituent of \mathcal{E} considered as an object of $\mathrm{Isoc}^{\dagger\dagger}(\mathcal{P}, X, Y)$ (without Frobenius). Then \mathcal{E}' automatically possesses a Frobenius structure, namely we have an isomorphism $\mathcal{E}' \xrightarrow{\sim} F^{*N}(\mathcal{E}')$ for some positive integer N .

Lemma 2.1.6. Let $\mathcal{E} \in F\text{-Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K)$.

1. The F -complex \mathcal{E} is ι -pure of weight w if and only if, for every closed point x of Y , $i_x^! (\mathcal{E})$ is ι -pure of weight w .
2. Let $f: (Y', X', \mathcal{P}') \rightarrow (Y, X, \mathcal{P})$ be a morphism of frames such that Y' is smooth as well. If \mathcal{E} is ι -mixed of weight $\leq w$ (resp. $\geq w$) then so are $f^+ \mathcal{E}$ and $f^! \mathcal{E}$.
3. If \mathcal{E} is ι -mixed of weight $\leq w$ (resp. $\geq w$) then so is $\mathbb{D}_{Y, \mathcal{P}}(\mathcal{E})$.
4. The notion of ι -mixedness is local in Y , namely \mathcal{E} is ι -mixed if and only if there exists an open covering $\{Y_i\}$ of Y such that $\mathcal{E}|_{Y_i}$ is ι -mixed for any i .

⁽⁷⁾We notice that there is a typo in the definition [Car06, 8.3.2]. Indeed, $w - d_{Y_i}$ should be replaced by $w + d_{Y_i}$.

Proof. The first statement follows from the isomorphism (1.2.13.1). The second and third ones are obvious. The last one is a consequence of the Lemma 1.4.6. \blacksquare

Lemma 2.1.7. *Let $0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0$ be an exact sequence in $F\text{-Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K)$. The overconvergent F -isocrystal \mathcal{E} is ι -mixed (resp. ι -pure of weight w , resp. ι -mixed of weight $\leq w$, resp. ι -mixed of weight $\geq w$) if and only if so are \mathcal{E}' and \mathcal{E}'' .*

Proof. Let $\text{gr}(\mathcal{E})$ be the direct sum of the constituents of \mathcal{E} , which is defined up to (non-canonical) isomorphism. We can check easily that $\text{gr}(\mathcal{E}) \xrightarrow{\sim} \text{gr}(\mathcal{E}') \oplus \text{gr}(\mathcal{E}'')$. \blacksquare

Corollary 2.1.8. 1. *Let $\mathcal{E} \in F\text{-Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K)$ and \mathcal{E}' be a subquotient of \mathcal{E} . If \mathcal{E} is ι -mixed (resp. ι -pure of weight w , resp. ι -mixed of weight $\leq w$, resp. ι -mixed of weight $\geq w$) then so is \mathcal{E}' .*

2. *Let $\mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}''$ be an exact sequence in $F\text{-Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K)$. If the overconvergent F -isocrystal \mathcal{E}' , \mathcal{E}'' are ι -mixed (resp. ι -pure of weight w , resp. ι -mixed of weight $\leq w$, resp. ι -mixed of weight $\geq w$) then so is \mathcal{E} .*

3. *If $\mathcal{E}_r^{i,j} \Rightarrow \mathcal{E}^{i+j}$ is a spectral sequence with $r \geq 1$, $i, j \geq 0$ where $\mathcal{E}_r^{i,j} \in F\text{-Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K)$ are ι -mixed. Then $\mathcal{E}^{i+j} \in F\text{-Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K)$ is ι -mixed.*

Proof. These are consequences of Lemma 2.1.7. \blacksquare

Definition 2.1.9. 1. We say that $\mathcal{E} \in F\text{-}D_{\text{isoc}}^b(Y, \mathcal{P}/K)$ is ι -mixed if $\mathcal{H}_t^j(\mathcal{E})$ is ι -mixed for any integer j . We denote by $F\text{-}D_{\text{isoc},m}^b(Y, \mathcal{P}/K)$ the full subcategory of $F\text{-}D_{\text{isoc}}^b(Y, \mathcal{P}/K)$ consisting of ι -mixed F -complexes.

2. We say that $\mathcal{E} \in F\text{-}D_{\text{isoc}}^b(Y, \mathcal{P}/K)$ is ι -pure of weight w (resp. ι -mixed of weight $\geq w$, resp. ι -mixed of weight $\leq w$) if $\mathcal{H}_t^j(\mathcal{E})$ is ι -pure of weight $w + j$ (resp. ι -mixed of weight $\geq w + j$, resp. ι -mixed of weight $\leq w + j$) for any integer j .

Remark 2.1.10. Assume Y is smooth, and let \mathcal{U} be an open formal subscheme of \mathcal{P} containing Y . Let $\mathcal{E} \in F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$. The property “ $\mathcal{E} \in F\text{-}D_{\text{isoc}}^b(Y, \mathcal{P}/K)$ and \mathcal{E} is ι -pure of weight w ” is equivalent to the property “ $\mathcal{E}|_{\mathcal{U}} \in F\text{-}D_{\text{isoc}}^b(Y, \mathcal{U}/K)$ and $\mathcal{E}|_{\mathcal{U}}$ is ι -pure of weight w ”. However, even if the property $\mathcal{E} \in F\text{-}D_{\text{isoc},m}^b(Y, \mathcal{P}/K)$ implies the property $\mathcal{E}|_{\mathcal{U}} \in F\text{-}D_{\text{isoc},m}^b(Y, \mathcal{U}/K)$, we do not know if the converse is true (see [Abe11, Remark 2.2]).

Remark 2.1.11. Suppose in this remark that X is proper. Let $\mathcal{E} \in F\text{-}D_{\text{isoc}}^b(Y, \mathcal{P}/K)$. Let U be a dense open subvariety of Y and $\mathcal{E}|_U \in F\text{-}D_{\text{isoc}}^b(U, \mathcal{P}/K)$ the restriction on $(U, \mathcal{P}/K)$ of \mathcal{E} . Then, it follows from [Cre98, 10.8] (or we can follow the p -adic analogue of the proof of the semicontinuity of weights of [KW01, I.2.8]) that \mathcal{E} is ι -pure of weight w if and only if $\mathcal{E}|_U$ is ι -pure of weight w . The hypothesis that X is proper is useful in this proof since we use the p -adic cohomological interpretation of the L -function. We do not know what is true when X is not proper.

Lemma 2.1.12. *Let $\mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \xrightarrow{+}$ be an exact triangle in $F\text{-}D_{\text{isoc}}^b(Y, \mathcal{P}/K)$. If the F -complexes \mathcal{E}' and \mathcal{E}'' are ι -mixed (resp. ι -pure of weight w , resp. ι -mixed of weight $\leq w$, resp. ι -mixed of weight $\geq w$), then so is \mathcal{E} .*

Proof. This is a consequence of Lemma 2.1.8.2. \blacksquare

2.2 Mixedness for overholonomic F -complexes

In this subsection, we define our one of the main players, ι -mixed F -complexes, and show some first properties. We continue to let (Y, X, \mathcal{P}) be a frame.

2.2.1. Recall from [Car08, 4.1.2] that an ordered set of subschemes $\{Y_i\}_{i=1,\dots,r}$ of Y is said to be a *smooth stratification* if the following holds: 1. $\{Y_i\}$ is a stratification, namely putting $Y_0 := \emptyset$, Y_k is an open subscheme of $Y \setminus \bigcup_{i < k} Y_i$ and $Y = \bigcup_{1 \leq i \leq r} Y_i$. 2. Y_i is a smooth variety. A stratification $\{Y'_j\}_{1 \leq j \leq r'}$ is said to be a *refinement of $\{Y_i\}$* if there exists an increasing function $\phi: [1, r] \cap \mathbb{N} \rightarrow [1, r'] \cap \mathbb{N}$ such that $\bigcup_{i \leq k} Y_i = \bigcup_{j \leq \phi(k)} Y'_j$ for any $1 \leq k \leq r$. The following facts are useful:

(*) For any stratification $\{Y_i\}_{1 \leq i \leq r}$ of Y , there is a smooth stratification $\{Y'_j\}_{1 \leq j \leq r'}$, which is a refinement of $\{Y_i\}$. Moreover, given two stratifications $\{Y_i\}$ and $\{Y'_j\}$, there exists a stratification $\{Y''_k\}$ which is a refinement of both $\{Y_i\}$ and $\{Y'_j\}$.

Definition 2.2.2. Let $\mathcal{E} \in F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$.

1. We say that \mathcal{E} is *ι -mixed* if there exists a smooth stratification $\{Y_i\}_{1 \leq i \leq r}$ of Y such that $\mathcal{E}|_{Y_i}$ is in $F\text{-}D_{\text{isoc}, m}^b(Y_i, \mathcal{P}/K)$ for any $i = 1, \dots, r$. We denote by $F\text{-}D_m^b(Y, \mathcal{P}/K)$ the full subcategory of $F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ consisting of ι -mixed F -complexes.
2. We say that \mathcal{E} is *ι -mixed of weight $\leq w$* if \mathcal{E} is ι -mixed and, for every closed point x in Y , the F -complex $i_x^+(\mathcal{E})$ is ι -mixed of weight $\leq w$. We denote by $F\text{-}D_{\leq w}^b(Y, \mathcal{P}/K)$ the full subcategory of $F\text{-}D_m^b(Y, \mathcal{P}/K)$ consisting of ι -mixed F -complexes of weight $\leq w$.
3. We say that \mathcal{E} is *ι -mixed of weight $\geq w$* if \mathcal{E} is ι -mixed and, for every closed point x in Y , the F -complex $i_x^!(\mathcal{E})$ is ι -mixed of weight $\geq w$. We denote by $F\text{-}D_{\geq w}^b(Y, \mathcal{P}/K)$ the full subcategory of $F\text{-}D_m^b(Y, \mathcal{P}/K)$ consisting of ι -mixed F -complexes of weight $\geq w$.
4. We say that \mathcal{E} is *ι -pure of weight w* if \mathcal{E} is both ι -mixed of weight $\leq w$ and of weight $\geq w$. The ι -weight of \mathcal{E} is denoted by $\text{wt}(\mathcal{E})$.

Remark 2.2.3. (i) Let $\mathcal{E} \in F\text{-}D_m^b(Y, \mathcal{P}/K)$. Then there exist real numbers w_1, w_2 such that \mathcal{E} is of weight both $\geq w_1$ and $\leq w_2$. Indeed, this is easy for overconvergent F -isocrystals and we reduce to this case by dévissage.

(ii) Let \mathcal{E} be an F -complex in $F\text{-}D_m^b(Y, \mathcal{P}/K)$. Assume that $\mathbb{D}_{Y, \mathcal{P}}(\mathcal{E})$ is ι -mixed as well. Then by definition, \mathcal{E} is ι -mixed of weight $\geq w$ (resp. $\leq w$) if and only if $\mathbb{D}_{Y, \mathcal{P}}(\mathcal{E})$ is ι -mixed of weight $\leq -w$ (resp. $\geq -w$). One of the main results of this paper is to show that, in fact, $\mathbb{D}_{Y, \mathcal{P}}(\mathcal{E})$ is ι -mixed if \mathcal{E} is. See Theorem 4.1.3 for the details.

(iii) Suppose that X is proper. Let $\mathcal{E} \in F\text{-}D_{\text{isoc}}^b(Y, \mathcal{P}/K)$. Then, it follows from the remark 2.1.11 that the F -complex \mathcal{E} is ι -mixed as an object of $F\text{-}D_{\text{isoc}}^b(Y, \mathcal{P}/K)$ if and only if \mathcal{E} is ι -mixed as an object of $F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$.

Proposition 2.2.4. Let $u: (Y', X', \mathcal{P}') \rightarrow (Y, X, \mathcal{P})$ be a morphism of frames, and let \mathcal{E} be in $F\text{-}D_{\geq w}^b(Y, \mathcal{P}/K)$. Then $u^!(\mathcal{E})$ is in $F\text{-}D_{\geq w}^b(Y', \mathcal{P}'/K)$.

Proof. Let $Y = \sqcup_{i=1,\dots,r} Y_i$ be a smooth stratification in P such that $\mathcal{E}|_{Y_i} \in F\text{-}D_{\text{isoc}, m}^b(Y_i, \mathcal{P}/K)$ for any $i = 1, \dots, r$. Let $b: Y' \rightarrow Y$ be the morphism defined by u . We get the stratification $Y' = \sqcup_{i=1,\dots,r} b^{-1}(Y_i)$. By using 2.2.1 (*), there exists a smooth stratification of Y' in P' which is a refinement of $Y' = \sqcup_{i=1,\dots,r} b^{-1}(Y_i)$. With Lemma 2.1.6.2, we can conclude. \blacksquare

Lemma 2.2.5. Let $u = (\text{id}, \star, \star, \star): (Y, X', \mathcal{P}', \mathcal{Q}') \rightarrow (Y, X, \mathcal{P}, \mathcal{Q})$ be a complete morphism of l.p. frames. Take $\mathcal{E} \in F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$, $\mathcal{E}' \in F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}'/K)$, and choose $\star \in \{m, \geq w, \leq w\}$.

1. Suppose Y to be smooth. We get $\mathcal{E} \in F\text{-}D_{\text{isoc}, \star}^b(Y, \mathcal{P}/K)$ if and only if $u^!(\mathcal{E}) \in F\text{-}D_{\text{isoc}, \star}^b(Y, \mathcal{P}'/K)$ (resp. $\mathcal{E}' \in F\text{-}D_{\text{isoc}, \star}^b(Y, \mathcal{P}'/K)$ if and only if $u_+(\mathcal{E}') \in F\text{-}D_{\text{isoc}, \star}^b(Y, \mathcal{P}/K)$).
2. We have $\mathcal{E} \in F\text{-}D_{\star}^b(Y, \mathcal{P}/K)$ if and only if $u^!(\mathcal{E}) \in F\text{-}D_{\star}^b(Y, \mathcal{P}'/K)$ (resp. $\mathcal{E}' \in F\text{-}D_{\star}^b(Y, \mathcal{P}'/K)$ if and only if $u_+(\mathcal{E}') \in F\text{-}D_{\star}^b(Y, \mathcal{P}/K)$).

Proof. From [Car12a, 2.5], the functors u_+ and $u^!$ induce canonical equivalence of categories between $F-D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ and $F-D_{\text{ovhol}}^b(Y, \mathcal{P}'/K)$. Hence, we reduce to check the part concerning the functor $u^!$. Moreover, from [Car12a, 2.5], we have the isomorphism $u^! \xrightarrow{\sim} u^+$. Then, by transitivity of the (extraordinary) inverse image, the part concerning “ $\leq w$ ” or “ $\geq w$ ” is a consequence of that concerning “ m ”. For the first statement, we may assume \mathcal{E} to be a convergent F -isocrystal, and show that \mathcal{E} is purely of weight w if and only if so is $u^!(\mathcal{E})$. For this, using Remark 2.1.10, we can suppose that T is empty, in which case the verification is easy and left to the reader.

Finally, let us prove the second one. The “only if” part follows from Proposition 2.2.4, so let us check the “if” part. Let $Y = \sqcup_{i=1, \dots, r} Y_i$ be a smooth stratification of Y such that for any $i = 1, \dots, r$, $u^!(\mathcal{E})|_{Y_i} \in F-D_{\text{isoc}, m}^b(Y_i, \mathcal{P}'/K)$. Since for any subvariety Y' of Y appearing in this stratification we have $\mathbb{R}\Gamma_{Y'}^\dagger(u^!(\mathcal{E})) \xrightarrow{\sim} u^!(\mathbb{R}\Gamma_{Y'}^\dagger(\mathcal{E}))$, we conclude. \blacksquare

Lemma 2.2.6. *Let $u = (j, a, g, f): (Y', X', \mathcal{P}', \mathcal{Q}') \rightarrow (Y, X, \mathcal{P}, \mathcal{Q})$ be a complete morphism of l.p. frames such that j is an immersion. If \mathcal{E}' is an F -complex in $F-D_m^b(Y', \mathcal{P}/K)$ (resp. $F-D_{\geq w}^b(Y', \mathcal{P}/K)$) then $u_+(\mathcal{E}')$ is an F -complex in $F-D_m^b(Y, \mathcal{P}/K)$ (resp. $F-D_{\geq w}^b(Y, \mathcal{P}/K)$).*

Proof. Let X'' be the closure of Y' in X , $c: X' \rightarrow X''$ and $d: X'' \rightarrow X$ the canonical factorization of a . We have $u = (j, d, \text{id}, \text{id}) \circ (\text{id}, c, g, f)$. By Lemma 2.2.5, we reduce to the case where $u = (j, d, \text{id}, \text{id})$. In that case $u_+(\mathcal{E}') = \mathcal{E}'$. Suppose that $Y' \subset Y$ is closed. Then we get the stratification $Y = (Y \setminus Y') \sqcup Y'$. Let $Y' = \sqcup_{i=1, \dots, r} Y'_i$ be a smooth stratification of Y' such that $\mathcal{E}'|_{Y'_i} \in F-D_{\text{isoc}, m}^b(Y'_i, \mathcal{P}/K)$ for any $i = 1, \dots, r$. Applying 2.2.1 (*) to the trivial stratification of $Y \setminus Y'$, there exist a smooth stratification $Y \setminus Y' = \sqcup_{i=1, \dots, s} Y''_i$. Thus, we get the smooth stratification $Y = (\sqcup_{i=1, \dots, s} Y''_i) \sqcup (\sqcup_{i=1, \dots, r} Y'_i)$. Since $\mathcal{E}'|_{Y''_i} = 0$, we get a desired stratification. Similarly, if $Y' \subset Y$ is open, then we get the stratification $Y = Y' \sqcup (Y \setminus Y')$ and we proceed as above by using 2.2.1 (*). \blacksquare

Proposition 2.2.7. *1. Let $\mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \xrightarrow{+}$ be a distinguished triangle in $F-D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$. If \mathcal{F}' and \mathcal{F}'' are ι -mixed (resp. of weight $\geq w$, resp. of weight $\leq w$), then so is \mathcal{F} .*

2. Let $\mathcal{E} \in F-D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$. Let $Y = \sqcup_{i=1, \dots, r} Y_i$ be a stratification of Y . The F -complex \mathcal{E} is ι -mixed (resp. ι -mixed of ι -weight $\geq w$) if and only if, for any $i = 1, \dots, r$, so are the F -complexes $\mathbb{R}\Gamma_{Y_i}^\dagger(\mathcal{E})$.

3. Let $\mathcal{E} \in F-D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$. The notion of ι -mixedness is local in Y , namely \mathcal{E} is ι -mixed if and only if there exists an open covering $\{Y_i\}$ of Y such that $\mathcal{E}|_{Y_i}$ is ι -mixed for any i .

Proof. Let us show the first claim. By 2.2.1 (*), there exists a smooth stratification $\{Y_i\}_{1 \leq i \leq r}$ of Y such that $\mathcal{F}'|_{Y_i}, \mathcal{F}|_{Y_i}, \mathcal{F}''|_{Y_i}$ are in $F-D_{\text{isoc}}^b(Y_i, \mathcal{P}/K)$ for any $i = 1, \dots, r$. We conclude thanks to Lemma 2.1.12. Now, let us show the second statement. By Proposition 2.2.4 and Lemma 2.2.6, if \mathcal{E} is a F -complex in $F-D_m^b(Y, \mathcal{P}/K)$ (resp. $F-D_{\geq w}^b(Y, \mathcal{P}/K)$) then so is $\mathbb{R}\Gamma_{Y_i}^\dagger(\mathcal{E}) = j_{i+} \circ j_i^!(\mathcal{E})$ where $j_i = (\star, \text{id}, \text{id}, \text{id}): (Y_i, X_i, \mathcal{P}, \mathcal{Q}) \rightarrow (Y, X, \mathcal{P}, \mathcal{Q})$ is the morphism of frames. Conversely, suppose that the F -complex $\mathbb{R}\Gamma_{Y_i}^\dagger(\mathcal{E})$ is in $F-D_m^b(Y, \mathcal{P}/K)$ (resp. $F-D_{\geq w}^b(Y, \mathcal{P}/K)$) for any $i = 1, \dots, r$. Then the second claim follows since \mathcal{E} possesses the same property by the following triangle:

$$\mathbb{R}\Gamma_{\sqcup_{j=i+1, \dots, r} Y_j}^\dagger(\mathcal{E}) \rightarrow \mathbb{R}\Gamma_{\sqcup_{j=i, \dots, r} Y_j}^\dagger(\mathcal{E}) \rightarrow \mathbb{R}\Gamma_{Y_i}^\dagger(\mathcal{E}) \xrightarrow{+},$$

and by using the first part of the proof. By using Lemma 2.1.6.4, the first and the second claim implies the third one. \blacksquare

Now, we show that the notion of ι -mixedness only depends on couple (Y, X) and not on the auxiliary choices.

Proposition 2.2.8. *Let $\ast \in \{m, \geq w, \leq w\}$.*

1. Let \mathbb{Y} be a couple and choose an l.p. frame $(Y, X, \mathcal{P}, \mathcal{Q})$ of \mathbb{Y} . The category $F-D_\ast^b(Y, \mathcal{P}/K)$ does not depend on the choice of the l.p. frame of \mathbb{Y} . We denote by $F-D_\ast^b(\mathbb{Y}/K)$ the corresponding category.

2. Let Y be a realizable variety, and choose an l.p. frame $(Y, X, \mathcal{P}, \mathcal{Q})$ of Y such that X is proper. The category $F\text{-}D_*^b(Y, \mathcal{P}/K)$ does not depend on the choice of the l.p. frame. We denote by $F\text{-}D_*^b(Y/K)$ the corresponding category.

Proof. Let us only prove the second one, since the first part of the proposition can be checked similarly. By using fiber products, we reduce to the case where there exists a complete morphism of l.p. frames $u: (Y, X', \mathcal{P}', \mathcal{Q}') \rightarrow (Y, X, \mathcal{P}, \mathcal{Q})$. In this case, the proposition is a consequence of Lemma 2.2.5. ■

Proposition 2.2.9. *Let \mathbb{Y} be a couple, and $\mathcal{E}, \mathcal{E}' \in F\text{-}D_{\text{ovhol}}^b(\mathbb{Y}/K)$. If \mathcal{E} is ι -mixed of weight $\geq w$ and \mathcal{E}' is ι -mixed of weight $\geq w'$ then $\mathcal{E} \otimes_{\mathbb{Y}} \mathcal{E}'$ is ι -mixed of weight $\geq w + w'$.*

Proof. By paragraph 1.1.9 (i), $\widetilde{\otimes}$ commutes with j_+ where j is a c-immersion. Thus, by dévissage, we may reduce to the case where Y is smooth and $\mathcal{E}, \mathcal{E}' \in F\text{-}D_{\text{isoc}, m}^b(Y, \mathcal{P}/K)$. In this isocrystal case, use paragraph 1.1.9 (i) again to show that the ι -mixedness is preserved and the estimation of weights. ■

Lemma 2.2.10. *Let $\mathcal{V} \rightarrow \mathcal{V}'$ be a finite homomorphism of complete discrete valuation ring with mixed characteristic $(0, p)$, and $k \rightarrow k'$ be the induced homomorphism of residue fields. Let K' be the field of fractions of \mathcal{V}' , and we put $X := X \otimes_k k'$, $Y' := Y \otimes_k k'$. Let $\mathcal{E} \in F\text{-}D_{\text{ovhol}}^b(Y, X/K)$ and \mathcal{E}' be the induced object of $F\text{-}D_{\text{ovhol}}^b(Y', X'/K')$. Then \mathcal{E} is ι -mixed (resp. of weight $\geq w$, resp. of weight $\leq w$) if and only if so is \mathcal{E}' .*

Proof. We leave the verification to the reader. ■

2.3 Estimation of weights in the curve case

In this section, we estimate the weights for curves. The proof is similar to that of Laumon in [Lau87], by using the methods developed in [AM11].

Definition 2.3.1. Let $\mathbb{Y} = (Y, X)$ be a couple such that Y is smooth, \mathcal{E} be an object of $F\text{-}\text{Isoc}^\dagger(\mathbb{Y}/K)$, and $i_y: (y, y) \rightarrow \mathbb{Y}$ be the canonical morphism for a closed point y of Y . The object \mathcal{E} is said to be ι -real if for any closed point y of Y , the characteristic polynomial $\iota \det(1 - t \cdot \text{Frob}_y; i_y^+(\mathcal{E}))$, which is *a priori* in $\mathbb{C}[t]$, is in $\mathbb{R}[t]$.

Theorem 2.3.2 ([Cre98], Theorem 10.5). *If an F -isocrystal \mathcal{E} on a smooth curve over k is ι -real, then any constituent of \mathcal{E} is ι -pure.*

Proof. It suffices to note that we did not assume \mathcal{E} to be quasi-unipotent contrary to the statement of [Cre98, 10.5] since the assumption is used only to assure the finiteness of \mathcal{E} and $\bigotimes^{2k} \mathcal{E}$ as written in *ibid.*, which is now known to be true if there is a Frobenius structure. ■

2.3.3. Let \mathcal{X} be a smooth formal curve over \mathcal{V} , and x be a closed point. For a holonomic module \mathcal{E} on \mathcal{X} , we denote by $\Psi_x(\mathcal{E})$ the nearby cycle $\Psi(\mathcal{E}|_{S_x})$ using the notation of [AM11, 2.1.5]. The theorem below follows directly from a result of Crew.

Theorem ([Cre98], Theorem 10.8). *Let \mathcal{X} be a smooth formal curve over \mathcal{V} , s be a closed point of \mathcal{X} , and we put $\mathcal{U} := \mathcal{X} \setminus \{s\}$. Let $j: \mathcal{U} \hookrightarrow \mathcal{X}$. Let \mathcal{E} be an overconvergent F -isocrystal ι -pure of weight w on \mathcal{U} .*

(i) Let α (resp. β) be an eigenvalue of the Frobenius acting on $\mathcal{H}_t^{-1}(i^+ j_+ \mathcal{E})$ (resp. $\mathcal{H}_t^0(i^+ j_+ \mathcal{E})$). Then we have the following estimations:

$$|\iota(\alpha)| \leq q^{\deg(s)(w-1)/2}, \quad |\iota(\beta)| \geq q^{\deg(s)(w+1)/2}$$

(ii) Let $s \in S$, and we put $\Psi := \Psi_s(j_+ \mathcal{E})$. Assume that the action of N on $\Psi/\Psi^{I=1, N=0}$ is trivial. Let α be the eigenvalue of the Frobenius acting on the latter quotient. Then $|\iota(\alpha)| = q^{\deg(s)w/2}$.

Proof. Let E be a differential F -module around s associated to \mathcal{E} . Then the weight of E is $w + 1$ by Remark 2.1.4, and [Cre98, Theorem 10.8] shows that $H_{\text{loc}}^0(E)$ is ι -weight $\leq w + 1$. We get (i) by applying Theorem 1.5.9 (iii) and duality.

The same theorem implies that $\text{gr}_i^M(\Psi(-1))$ is purely of weight $w + 1 + i$. Under the situation of (ii), $\Psi/\Psi^{I=1, N=0}$ is nothing but $\text{gr}_1^M(\Psi)$, and we get what we wanted. ■

2.3.4. Let us briefly review the geometric Fourier transform [NH04], only in the affine space case.

Fix $\pi \in \overline{\mathbb{Q}}_p$ such that $\pi^{p-1} = -p$. We assume that $\pi \in K$. We denote by \mathcal{L}_π the Artin-Schreier isocrystal in $F\text{-Isoc}^{\dagger\dagger}(\mathbb{A}_k^1/K)$. Since \mathcal{L}_π is a direct factor of $\text{Art}_* \mathcal{O}_{\mathbb{A}_k^1}$ (cf. [Ber84, 1.5]), where $\text{Art}: \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ is the Artin-Schreier morphism, the weight of $\mathcal{L}_\pi[1]$ is 0.

Now, consider the following diagram: $(\mathbb{A}_k^n)' \xleftarrow{p_2} \mathbb{A}_k^{2n} \xrightarrow{p_1} \mathbb{A}_k^n$, where $(\mathbb{A}_k^n)'$ is the “dual affine space”, which is nothing but \mathbb{A}_k^n . We denote by $\delta: \mathbb{A}_k^n \times (\mathbb{A}_k^n)' \rightarrow \mathbb{A}_k^1$ the canonical duality bracket induced by $t \mapsto \sum_{i=1}^n x_i y_i$. We put $\mathcal{K}_\pi := \delta^!(\mathcal{L}_\pi[-1])$. For any $\mathcal{E} \in F\text{-}D_{\text{ovhol}}^b(\mathbb{A}_k^n)$, the geometric Fourier transform $\mathcal{F}_\pi(\mathcal{E})$ is defined to be $p_{2+}(p_1^! \mathcal{E} \otimes_{\mathbb{A}_k^{2n}} \mathcal{K}_\pi)$ (cf. [NH04, 3.2.1]⁽⁹⁾). Important properties for us are that:

1. if \mathcal{E} is a F -module, $\mathcal{F}_\pi(\mathcal{E})[2 - n]$ is a F -module as well (cf. [NH04, Theorem 5.3.1]).
2. $\mathcal{F}'_\pi \circ \mathcal{F}_\pi(\mathcal{E})[4 - 2n] \cong \mathcal{E}(n)$ where \mathcal{F}'_π denotes the geometric Fourier transform for dual affine space (cf. [AM11, Remark 3.2.8]).

2.3.5. In this subsection, we use the Fourier transform for $n = 1$. Let $\mathbb{A} := \mathbb{A}^1$. Consider the following three types of *irreducible* overholonomic F -modules \mathcal{E} on \mathbb{A} :

- (T1) Let s be a closed point of \mathbb{A} , and $i_s: s \hookrightarrow \mathbb{A}$. Then $\mathcal{E} \cong i_{s+} V$ where V is an irreducible K -vector space with Frobenius.
- (T2) There exists a simple overholonomic F -module \mathcal{E}' on \mathbb{A}' of type (T1) such that $\mathcal{E} \cong \mathcal{F}_\pi(\mathcal{E}')$.
- (T3) We have $\mathcal{E} \cong j_{!+} \mathcal{F}$ where $j: \mathbb{A} \setminus S \hookrightarrow \mathbb{A}$ with S a finite subset of \mathbb{A} and \mathcal{F} is an irreducible F -isocrystal which is not of type (T2).

These F -modules are irreducible, since Fourier transform is involutive by 2.3.4.2 and by Proposition 1.4.7. Moreover, irreducible objects can be classified in the above three types. By definition, F -modules of type (T1) are transformed under Fourier transform to (T2) and (T2) to (T1). Thus F -modules of type (T3) are transformed under Fourier transform to (T3).

Lemma 2.3.6. *Let $p: \mathbb{A} := \mathbb{A}^1 \rightarrow \text{Spec}(k)$ be the structural morphism, S be a set of closed points in \mathbb{A} , and $j: U := \mathbb{A} \setminus S \hookrightarrow \mathbb{A}$ be the canonical inclusion. Let \mathcal{E} be an overconvergent F -isocrystal purely of ι -weight w on U . We assume moreover that \mathcal{E} is irreducible, unramified at ∞ , and not geometrically constant. Then for any eigenvalue α of the Frobenius acting on $H^0(p_+ j_+ \mathcal{E})$, we have the estimation $|\iota(\alpha)| \geq q^{w/2}$.*

Proof. Since we may change K by its totally ramified extension by Lemma 2.2.10, we may assume and fix a root π of the equation $x^{p-1} + p = 0$ in K . By [AM11, Proposition 4.1.6 (ii)], $0' \in \mathbb{A}'$ is the only singularity of $\mathcal{F}_\pi(\mathcal{E})$. By assumption, \mathcal{E} is a simple F -module of type (T3). This is showing that $\mathcal{F}_\pi(\mathcal{E})$ is of type (T3) as well, and thus there exists an irreducible overconvergent F -isocrystal \mathcal{E}' on $\mathbb{A}' \setminus \{0'\}$ such that $\mathcal{F}_\pi(j_{!+} \mathcal{E}[1]) \cong j'_{!+}(\mathcal{E}')$ where $j': \mathbb{A}' \setminus \{0'\} \hookrightarrow \mathbb{A}'$. By duality, it suffices to show that any eigenvalue of $\mathcal{H}_t^0(p_! j_! \mathcal{E})$ is of weight $\leq w$, and by Theorem 2.3.3 (i), we are reduced to showing that $\mathcal{H}_t^0(p_! j_{!+} \mathcal{E})$ is of weight $\leq w$.

In this proof, for the notation being compatible with [AM11], we introduce the following: for a smooth curve $q: X \rightarrow \text{Spec}(k')$ where k' is a field over k and a overholonomic F -module \mathcal{F} on X , we denote $\mathcal{H}_t^i(q_! \mathcal{F})$

⁽⁸⁾There is a typo in the definition of \mathcal{K}_π in [NH04, 3.1.1]: the shift of \mathcal{K}_π is not $[2 - 2N]$ but $[2N - 2]$.

⁽⁹⁾Notice that our twisted tensor product and hers are the same.

by $H_{\text{rig},c}^{i+1}(X, \mathcal{F})(1)$. Using this notation, we need to show that $H_{\text{rig},c}^1(\mathbb{A}, j_{!+}\mathcal{E})(1)$ is of weight $\leq w$. By the same argument as [AM11, 6.2.8], we have an exact sequence of Deligne modules

$$0 \rightarrow H_{\text{rig},c}^1(\mathbb{A}_{\bar{k}}, j_{!+}\mathcal{E}) \rightarrow \Psi_{0'}((j_{!+}\mathcal{E})^\wedge|_{S_{0'}})(-2) \rightarrow (K^{\text{ur}} \otimes_K \mathcal{E}|_{s_\infty})(-1) \rightarrow \underbrace{H_{\text{rig},c}^2(\mathbb{A}_{\bar{k}}, j_{!+}\mathcal{E})}_{=0} \rightarrow 0.$$

Moreover, we have

$$(\Psi_{0'}((j_{!+}\mathcal{E})^\wedge)(-2))^{I=1, N=0} \cong \mathcal{H}_t^{-1}(i_{0'}^+ j_{!+}'(\mathcal{E}'(-2))) \cong \mathcal{H}_t^0(p_! j_{!+}\mathcal{E}(-1)) \cong H_{\text{rig},c}^1(\mathbb{A}_{\bar{k}}, j_{!+}\mathcal{E}) \quad (\star)$$

where I denotes the inertia group at $0'$. The first isomorphism is deduced by Theorem 1.5.9 (iii).

Now, assume \mathcal{E}' is ι -pure of weight $w' \in \mathbb{R}$. Then Theorem 2.3.3 (ii) can be applied to get

$$\Psi_{0'}(\mathcal{E}')(-2)/(\Psi_{0'}(\mathcal{E}')(-2))^{I=1, N=0} \cong (K^{\text{ur}} \otimes_K \mathcal{E}|_{s_\infty})(-1)$$

is ι -pure of weight $w' + 4$. Applying Theorem 2.3.3 (i) to \mathcal{E} and $\mathbb{D}(\mathcal{E})$, $(K^{\text{ur}} \otimes_K \mathcal{E}|_{s_\infty})$ is ι -pure of ι -weight $w + 1$, which implies that $w' = w - 1$. Applying Theorem 2.3.3 (i) to the isomorphisms (\star) , we get the desired upper bound.

It remains to show that \mathcal{E}' is ι -pure. Using Theorem 2.3.2, it suffices to find an ι -real overconvergent F -isocrystal \mathcal{G}' on $\mathbb{A}^1 \setminus \{0'\}$ of which \mathcal{E}' is a constituent. For this, we just follow exactly the same line as the last half part of [Lau87, (4.4)], and we omit the detail here. \blacksquare

Theorem 2.3.7. *Let $f: X \rightarrow \text{Spec}(k)$ be a smooth curve, and \mathcal{E} be an object of $F\text{-}D_{\geq w}^b(X/K)$. Then $f_+\mathcal{E}$ is of weight $\geq w$.*

Proof. By dévissage, we may assume \mathcal{E} to be an overconvergent F -isocrystal on X . Let Z be a closed subscheme of X . By Proposition 2.2.7, $\mathbb{R}\Gamma_Z^\dagger(\mathcal{E})$ is ι -mixed of weight $\geq w$. Thus by considering the localization triangle, we may replace X by $X \setminus Z$, and shrink X . Since, by shrinking X , there exists a non-constant morphism $g: X \xrightarrow{g} X' \subset \mathbb{P}_k^1$ such that g is finite étale, we may assume that $X \subset \mathbb{P}^1$. Since it suffices to show the theorem after taking a finite extension of k , we may assume that $\infty \in \mathbb{P}^1$ is contained in X . Let $\tilde{\mathcal{E}}$ be the push-forward of \mathcal{E} to \mathbb{P}^1 . We put $j: \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ and $i: \{\infty\} \hookrightarrow \mathbb{P}^1$. Consider the following exact sequence:

$$\mathbb{R}\Gamma_{\{\infty\}}^\dagger(\mathcal{E}) \rightarrow \tilde{\mathcal{E}} \rightarrow j_+(\tilde{\mathcal{E}}|_{\mathbb{A}^1}) \xrightarrow{+1}.$$

Since $\mathbb{R}\Gamma_{\{\infty\}}^\dagger(\mathcal{E})$ is ι -mixed of weight $\geq w$, it is reduced to showing that $p_+(\tilde{\mathcal{E}}|_{\mathbb{A}^1})$ is of weight $\geq w$ where $p: \mathbb{A}^1 \rightarrow \text{Spec}(k)$ is the structural morphism. We may assume \mathcal{E} to be irreducible. When \mathcal{E} is not geometrically constant, then the theorem follows by Lemma 2.3.6. Otherwise, the verification is easy. \blacksquare

Remark. In fact, this theorem is not completely new, and by Remark 2.1.4, it follows directly from the main theorem of [Ked06]. However, the proof is independent.

3 Monodromy filtration of a convergent log-isocrystal

Throughout this section, we consider situation (B) in Notation and convention. Before starting the section, let us fix notation.

3.1 Notation

Until §3.6, we consider the following situation unless otherwise stated.

3.1.1. Let \mathcal{X} be a smooth formal \mathcal{V} -scheme with local coordinates denoted by t_1, \dots, t_d . For any $i = 1, \dots, d$, we put $\mathcal{Z}_i = V(t_i)$. We denote by $B := A/t_1 A$, so $\mathcal{Z}_1 := \mathrm{Spf} B$. Fix $1 \leq r \leq d$, and we denote by \mathfrak{D} the strict normal crossing divisor of \mathcal{X} whose irreducible components are $\mathcal{Z}_2, \dots, \mathcal{Z}_r$ and $\mathfrak{D} = \emptyset$ if $r = 1$. Put $\mathcal{Z} := \mathcal{Z}_1 \cup \mathfrak{D}$. We get a strict normal crossing divisor of \mathcal{Z}_1 defined by $\mathfrak{D}_1 := \bigcup_{i=2}^r \mathcal{Z}_1 \cap \mathcal{Z}_i$. We put $\mathcal{Y} := \mathcal{X} \setminus \mathcal{Z}$, and let $j: \mathcal{Y} \subset \mathcal{X}$ be the open immersion, $u: (\mathcal{X}, \mathcal{Z}) \rightarrow \mathcal{X}$, $u_1: (\mathcal{X}, \mathcal{Z}) \rightarrow (\mathcal{X}, \mathfrak{D})$, $v: (\mathcal{X}, \mathfrak{D}) \rightarrow \mathcal{X}$, $i_1: \mathcal{Z}_1 \hookrightarrow \mathcal{X}$, $i_1^\# : (\mathcal{Z}_1, \mathfrak{D}_1) \rightarrow (\mathcal{X}, \mathfrak{D})$, $w_1: (\mathcal{Z}_1, \mathfrak{D}_1) \rightarrow \mathcal{Z}_1$ be the canonical morphisms of *log schemes*. We sometimes denote $\mathcal{X}^b := (\mathcal{X}, \mathcal{Z})$, $\mathcal{X}^\# := (\mathcal{X}, \mathfrak{D})$, $\mathcal{Z}_1^\# := (\mathcal{Z}_1, \mathfrak{D}_1)$ for simplicity.

3.1.2. Local coordinates of paragraph 3.1.1 induce canonically the commutative diagram:

$$\begin{array}{ccc} \mathcal{Z}_1 & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \widehat{\mathbb{A}}^{n-1} & \longrightarrow & \widehat{\mathbb{A}}^n \end{array}$$

where the vertical morphisms are étale. The coordinate t_1 induces the closed immersion $\widehat{\mathbb{A}}^1 \hookrightarrow \widehat{\mathbb{A}}^n$, and we have an étale morphism $\mathcal{Z}_1 \times \widehat{\mathbb{A}}^1 \rightarrow \widehat{\mathbb{A}}^n$. By putting $\mathcal{X}' := (\mathcal{Z}_1 \times \widehat{\mathbb{A}}^1) \times_{\widehat{\mathbb{A}}^n} \mathcal{X}$, which is étale over \mathcal{X} , there is a section $\mathcal{Z}_1 \rightarrow \mathcal{X}'$ of the smooth projection $g_1: \mathcal{X}' \rightarrow \mathcal{Z}_1$. By shrinking \mathcal{X}' but not \mathcal{X} , we may assume that $\mathcal{Z}_1 \xrightarrow{\sim} \mathcal{Z}_1 \times_{\mathcal{X}} \mathcal{X}'$ by SGA1 Exp. I Corollary 5.3. We put $\mathfrak{D}' := f^{-1}(\mathfrak{D})$, $\mathcal{Z}' := f^{-1}(\mathcal{Z})$, $\mathcal{X}^b := (\mathcal{X}', \mathcal{Z}')$, $\mathcal{X}'^\# := (\mathcal{X}', \mathfrak{D}')$. We denote by $g_1^b: \mathcal{X}^b \rightarrow \mathcal{Z}_1^\#$, $g_1^\#: \mathcal{X}'^\# \rightarrow \mathcal{Z}_1^\#$, $i_1'^\# : \mathcal{Z}_1^\# \rightarrow \mathcal{X}'^\#$, $v': \mathcal{X}'^\# \rightarrow \mathcal{X}'$ the associated morphisms. We summarize and introduce some notation in the following diagrams:

$$\begin{array}{ccccc} \mathcal{Z}_1 & \xrightleftharpoons{g_1} & \mathcal{X}' & \xleftarrow{\quad} & \mathcal{X}'^\# & \xrightleftharpoons{g_1^\#} & \mathcal{Z}_1^\# \\ \parallel & \square & \downarrow f & \square & \downarrow f^\# & \square & \parallel \\ \mathcal{Z}_1 & \xrightarrow{i_1} & \mathcal{X} & \xleftarrow{v} & \mathcal{X}^\# & \xleftarrow{i_1^\#} & \mathcal{Z}_1^\# \end{array} \quad \begin{array}{ccc} \mathcal{X}'^\# & \xleftarrow{u_1'} & \mathcal{X}^b \\ f^\# \downarrow & \square & \downarrow f^b \\ \mathcal{X}^\# & \xleftarrow{u_1} & \mathcal{X}^b \end{array} \quad (3.1.2.1)$$

where squares are cartesian by definition (we also notice that morphisms are exact). We note that $\mathfrak{D}' = g_1^{-1} \circ i_1'^{-1}(\mathfrak{D}')$.

Let $F: \widehat{\mathbb{A}}_{\mathcal{V}}^d \rightarrow \widehat{\mathbb{A}}_{\mathcal{V}}^d$ and $F: \widehat{\mathbb{A}}_{\mathcal{V}}^{d-1} \rightarrow \widehat{\mathbb{A}}_{\mathcal{V}}^{d-1}$ be the canonical Frobenius endomorphisms $P(\underline{X}) \mapsto P(\underline{X}^q)$. Since the morphism $\mathcal{X} \rightarrow \widehat{\mathbb{A}}_{\mathcal{V}}^d$ (resp. $\mathcal{Z}_1 \rightarrow \widehat{\mathbb{A}}_{\mathcal{V}}^{d-1}$) induced by the local coordinates t_1, \dots, t_d (resp. t_2, \dots, t_d) is étale, the Frobenius endomorphism of X (resp. of Z) has a unique lifting $F_{\mathcal{X}}: \mathcal{X} \rightarrow \mathcal{X}$, (resp. $F_{\mathcal{Z}_1}: \mathcal{Z}_1 \rightarrow \mathcal{Z}_1$) which commutes with the canonical Frobenius endomorphism of $\widehat{\mathbb{A}}_{\mathcal{V}}^d$ (resp. $\widehat{\mathbb{A}}_{\mathcal{V}}^{d-1}$). Since the local coordinates t_1, \dots, t_d are fixed we can call abusively these Frobenius endomorphisms “canonical”.

3.1.3. Let \mathcal{F} be a coherent $\mathcal{D}_{\mathcal{X}^b, \mathbb{Q}}^\dagger$ -module, projective of finite type over $\mathcal{O}_{\mathcal{X}, \mathbb{Q}}$ with nilpotent residues. We put $\mathcal{E} = u_+(\mathcal{F})$. We suppose that \mathcal{E} is endowed with Frobenius structure $\phi_{\mathcal{E}}: F_{\mathcal{X}}^*(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$. Since $\mathcal{E} \xrightarrow{\sim} \mathcal{F}(\dagger Z)$, from the equivalence of categories of [Ked07, 6.4.5], there exist a unique isomorphism $\phi_{\mathcal{F}}: F_{\mathcal{X}}^*(\mathcal{F}) \xrightarrow{\sim} \mathcal{F}$ such that $(\dagger Z)(\phi_{\mathcal{F}}) = \phi_{\mathcal{E}}$.

3.2 Monodromy filtration of a convergent log-isocrystal with nilpotent residues

We keep the notation of §3.1.

Definition 3.2.1. We define the category \mathfrak{C} as follows. An object is a coherent $F\text{-}\mathcal{D}_{\mathcal{Z}_1^\#, \mathbb{Q}}^\dagger$ -module V which is locally projective over $\mathcal{O}_{\mathcal{Z}_1, \mathbb{Q}}$ with nilpotent residues and endowed with an $F\text{-}\mathcal{D}_{\mathcal{Z}_1^\#, \mathbb{Q}}^\dagger$ -linear morphism $N: V(1) \rightarrow V$ so that N is nilpotent if we forget Frobenius structures. A morphism $(V, N) \rightarrow (V', N')$ is an $F\text{-}\mathcal{D}_{\mathcal{Z}_1^\#, \mathbb{Q}}^\dagger$ -linear morphism $V \rightarrow V'$ commuting with the nilpotent endomorphisms. We denote by \mathfrak{C}^0 the full subcategory of \mathfrak{C} of objects (V, N) such that $N = 0$. We check easily that both categories \mathfrak{C} and \mathfrak{C}^0 are abelian.

Definition 3.2.2. Let $(V, N), (V', N') \in \mathfrak{C}$. We define the following operations in \mathfrak{C} :

1. We put $V^\vee := \mathcal{H}om_{\mathcal{O}_{\mathbb{Z}_1, \mathbb{Q}}}(V, \mathcal{O}_{\mathbb{Z}_1, \mathbb{Q}})$. Let $N^\vee: V^\vee \rightarrow V(1)^\vee$ be the morphism defined by $\phi \mapsto -\phi \circ N$. Since $V(1)^\vee = V^\vee(-1)$, $(V^\vee, N^\vee(-1))$ is an object in \mathfrak{C} , and denoted by $(V, N)^\vee$ called the *dual of* (V, N) .
2. The *tensor product* of (V, N) and (V', N') in \mathfrak{C} is $(V, N) \otimes_{\mathcal{O}_{\mathbb{Z}_1, \mathbb{Q}}}(V', N') := (V \otimes_{\mathcal{O}_{\mathbb{Z}_1, \mathbb{Q}}} V', N \otimes \text{id} + \text{id} \otimes N')$.
3. For any $\phi \in \mathcal{H}om_{\mathcal{O}_{\mathbb{Z}_1, \mathbb{Q}}}(V, V')(1)$, we define $\text{Hom}(N, N')(\phi) := N' \circ \phi - \phi \circ N(-1)$. We define the *internal Hom* by $\mathcal{H}om_{\mathcal{O}_{\mathbb{Z}_1, \mathbb{Q}}}((V, N), (V', N')) := (\mathcal{H}om_{\mathcal{O}_{\mathbb{Z}_1, \mathbb{Q}}}(V, V'), \text{Hom}(N, N'))$.

Remark 3.2.3. Let $(V, N), (V', N') \in \mathfrak{C}$. Since V is a projective $\mathcal{O}_{\mathbb{Z}_1, \mathbb{Q}}$ -module the canonical $\mathcal{O}_{\mathbb{Z}_1, \mathbb{Q}}$ -linear morphism $V^\vee \otimes_{\mathcal{O}_{\mathbb{Z}_1, \mathbb{Q}}} V' \rightarrow \mathcal{H}om_{\mathcal{O}_{\mathbb{Z}_1, \mathbb{Q}}}(V, V')$ is an isomorphism. We can show that this isomorphism commutes with the canonical nilpotent endomorphisms, and induces an isomorphism in \mathfrak{C} .

Lemma 3.2.4. *Let $(V, N) \in \mathfrak{C}$. Then, the evaluation homomorphism induces the canonical isomorphism: $\text{ev}: (V, N) \xrightarrow{\sim} ((V, N)^\vee)^\vee$. Moreover, we get the canonical isomorphisms:*

$$(\ker N(-1))^\vee \xrightarrow{\sim} \text{coker}(N^\vee(-1)), \quad (\text{coker } N)^\vee \xrightarrow{\sim} \ker(N^\vee). \quad (3.2.4.1)$$

Proof. The proof is straightforward. ■

3.2.5 (Monodromy filtration). Let $(V, N) \in \mathfrak{C}$. As in the proof of [Del80, 1.6.1], there exists a unique finite increasing filtration M on V such that $NM_i \subset M_{i-2}(-1)$, and N^k induces an isomorphism $\text{gr}_k^M(V) \xrightarrow{\sim} \text{gr}_{-k}^M(V)(-k)$. We call this filtration the *monodromy filtration* and it will usually be denoted by M . By construction (cf. the proof of [Del80, 1.6.1]), the induced filtration on $\ker N(-1)$ is such that $\text{gr}_i^M(\ker N(-1)) = 0$ for $i > 0$. By duality lemma 3.2.4, we get a filtration on $\text{coker } N$ such that $\text{gr}_i^M \text{coker } N = 0$ if $i < 0$.

Remark 3.2.6. Let ϕ be the forgetful functor from the category of $F\mathcal{D}_{\mathbb{Z}_1^\#, \mathbb{Q}}^\dagger$ -modules to the category of $\mathcal{D}_{\mathbb{Z}_1^\#, \mathbb{Q}}^\dagger$ -modules. Let \mathfrak{D} be the category defined in the same way as \mathfrak{C} but without Frobenius structure and let $\phi: \mathfrak{C} \rightarrow \mathfrak{D}$ be the forgetful functor defined by $\phi(V, N) = (\phi(V), \phi(N))$. For any $(V, N) \in \mathfrak{C}$, we have a monodromy filtration on $\phi(V, N)$ denoted by M' . Then by the uniqueness property of monodromy filtration we get that $\phi(M_i) \cong M'_i$.

3.2.7. Let $(V, N), (V', N'), (V'', N'') \in \mathfrak{C}$. As in [Del80, 1.6.9], we have

$$M_i((V, N)^\vee) = M_{-i-1}(V, N)^\perp, \quad M_i((V', N') \otimes (V'', N'')) = \sum_{i'+i''=i} M_{i'}(V', N') \otimes M_{i''}(V'', N'').$$

We have

$$\text{gr}_i^M((V, N)^\vee) = \text{gr}_i^M(V, N)^\perp, \quad \text{gr}_i^M((V', N') \otimes (V'', N'')) = \sum_{i'+i''=i} \text{gr}_{i'}^M(V', N') \otimes \text{gr}_{i''}^M(V'', N'').$$

3.2.8 (Monodromy filtrations coming from \mathcal{F}). We put $\mathcal{H} := i_1^*(\mathcal{F})$. We remark that the action of $t_1\partial_1$ on \mathcal{F} induces the residue morphism $N_{1, \mathcal{F}}: \mathcal{H} \rightarrow \mathcal{H}$, which is a $\mathcal{D}_{\mathbb{Z}_1^\#, \mathbb{Q}}^\dagger$ -linear homomorphism and nilpotent by hypothesis on \mathcal{F} . Let $f \in \mathcal{O}_{\mathcal{X}}$ and $m \in \mathcal{F}$ so that $f \otimes m \in F_{\mathcal{X}}^*(\mathcal{F}) \subset F_{\mathcal{X}}^*(\mathcal{F}(\dagger Z))$. For any $1 \leq i \leq d$, we have

$$t_i\partial_i(f \otimes m) = t_i\partial_i(f) \otimes m + f \otimes qt_i\partial_i(m) \quad (3.2.8.1)$$

in $F_{\mathcal{X}}^*(\mathcal{F})$. This computation shows that the left square of the following diagram is commutative:

$$\begin{array}{ccccc} F_{\mathbb{Z}_1}^* \mathcal{H} & \xrightarrow{\sim} & i_1^* F_{\mathcal{X}}^*(\mathcal{F}) & \xrightarrow{i_1^*(\phi_{\mathcal{F}})} & i_1^*(\mathcal{F}) \\ qF_{\mathbb{Z}_1}^*(N_{1, \mathcal{F}}) \downarrow & & N_{1, F_{\mathcal{X}}^* \mathcal{F}} \downarrow & & \downarrow N_{1, \mathcal{F}} \\ F_{\mathbb{Z}_1}^* \mathcal{H} & \xrightarrow{\sim} & i_1^* F_{\mathcal{X}}^*(\mathcal{F}) & \xrightarrow{i_1^*(\phi_{\mathcal{F}})} & i_1^*(\mathcal{F}). \end{array} \quad (3.2.8.2)$$

Since the right square is commutative by functoriality, the diagram (3.2.8.2) is commutative. Now, we get the Frobenius structure $\phi_{\mathcal{H}}: F_{\mathcal{Z}_1}^* \mathcal{H} \xrightarrow{\sim} i_1^* F_{\mathcal{X}}^*(\mathcal{F}) \xrightarrow{i_1^* \phi_{\mathcal{F}}} i_1^*(\mathcal{F}) = \mathcal{H}$ on \mathcal{H} . The commutativity of the diagram (3.2.8.2) shows that the homomorphism $N_{1,\mathcal{F}}: \mathcal{H}(1) \rightarrow \mathcal{H}$ commutes with Frobenius. Since \mathcal{F} has nilpotent residues, we get $(\mathcal{H}, N_{1,\mathcal{F}}) \in \mathfrak{C}$. Since the functor $(\dagger \mathfrak{D}_1)$ is exact, we get a filtration on $(\ker N_{1,\mathcal{F}}(-1))(\dagger \mathfrak{D}_1)$ and another on $(\operatorname{coker} N_{1,\mathcal{F}})(\dagger \mathfrak{D}_1)$. By abuse of language, these filtrations are also called the *monodromy filtrations*.

3.3 Comparison with Crew's Frobenius structure

3.3.1 (Crew's Frobenius structure). Using the notation of §3.1, suppose in this paragraph that X is of dimension 1 and that Z is k -rational. We put $F := \Gamma(\mathcal{X}, \mathcal{F})$, $F_{|Z|} := \Gamma(\mathcal{I}Z_{|\mathcal{X}|}, \operatorname{sp}^*(\mathcal{E}))$ and $E_{|Z|} := \Gamma(\mathcal{I}Z_{|\mathcal{X}|}, \operatorname{sp}^*(\mathcal{E}))$. We denote by i_1^* the cokernel of the multiplication by t_1 , $F_{|Z|}^{\nabla\infty} := \bigcup_{n>0} \ker((t_1 \partial_1)^n: F_{|Z|} \rightarrow F_{|Z|})$ and $E_{|Z|}^{\nabla\infty} := \bigcup_{n>0} \ker((t_1 \partial_1)^n: E_{|Z|} \rightarrow E_{|Z|})$. To simplify notation, we denote by ϕ the Frobenius structure on $F, F_{|Z|}, E_{|Z|}$, which are induced by extension from that on F .

Since the canonical Frobenius endomorphism $F^*: \mathcal{R}_K \rightarrow \mathcal{R}_K$ (the local coordinate t_1 is fixed) is K -linear, we have $F^*(\mathcal{R}_K \otimes_K E_{|Z|}^{\nabla\infty}) \xrightarrow{\sim} \mathcal{R}_K \otimes_K F^*(E_{|Z|}^{\nabla\infty}) = \mathcal{R}_K \otimes_K E_{|Z|}^{\nabla\infty}$. Similarly, we get $F^*(\mathcal{O}_K^{\text{an}} \otimes_K F_{|Z|}^{\nabla\infty}) \xrightarrow{\sim} \mathcal{O}_K^{\text{an}} \otimes_K F_{|Z|}^{\nabla\infty}$. We define ψ_{η} and ψ so that the corresponding squares of the diagram

$$\begin{array}{ccccccc}
 F_{|Z|}^{\nabla\infty} & \xrightarrow{\quad} & \mathcal{O}_K^{\text{an}} \otimes_K F_{|Z|}^{\nabla\infty} & \xrightarrow{F^*(\text{can})} & F^*(F_{|Z|}) & \xrightarrow{\pi_1^*} & F_{|Z|}/t_1 F_{|Z|} \\
 \downarrow \exists! \psi_{\eta} & \swarrow & \downarrow \exists! \psi & \swarrow & \downarrow \phi & \swarrow & \downarrow i_1^*(\phi) \\
 E_{|Z|}^{\nabla\infty} & \xrightarrow{\quad} & \mathcal{R}_K \otimes_K E_{|Z|}^{\nabla\infty} & \xrightarrow{F^*(\text{can})} & F^*(E_{|Z|}) & \xrightarrow{\pi_1^*} & F_{|Z|}/t_1 F_{|Z|} \\
 \downarrow \exists! \psi_{\eta} & \swarrow & \downarrow \exists! \psi & \swarrow & \downarrow \phi & \swarrow & \downarrow i_1^*(\phi) \\
 F_{|Z|}^{\nabla\infty} & \xrightarrow{\quad} & \mathcal{O}_K^{\text{an}} \otimes_K F_{|Z|}^{\nabla\infty} & \xrightarrow{\text{can}} & F_{|Z|} & \xrightarrow{\pi_1^*} & F_{|Z|}/t_1 F_{|Z|} \\
 \downarrow \exists! \psi_{\eta} & \swarrow & \downarrow \exists! \psi & \swarrow & \downarrow \phi & \swarrow & \downarrow i_1^*(\phi) \\
 E_{|Z|}^{\nabla\infty} & \xrightarrow{\quad} & \mathcal{R}_K \otimes_K E_{|Z|}^{\nabla\infty} & \xrightarrow{\text{can}} & E_{|Z|} & \xrightarrow{\pi_1^*} & F_{|Z|}/t_1 F_{|Z|}
 \end{array} \tag{3.3.1.1}$$

where π_1^* is the canonical projection, are commutative. We may check that this diagram is commutative. The isomorphism $E_{|Z|}^{\nabla\infty} \xrightarrow{\sim} E_{|Z|}$ that we get is Crew's Frobenius action defined in [Cre98, §10].

3.3.2 (Comparison with Crew's Frobenius structure). We keep in this paragraph the hypotheses and notation of paragraph 3.3.1. Following [Ked07, 3.6.2] (resp. [Ked07, 3.6.9]), $F_{|Z|}$ (resp. $E_{|Z|}$) is unipotent. Then, we get that $\mathcal{R}_K \otimes_K^{\text{an}} F_{|Z|} \rightarrow E_{|Z|}$ is an isomorphism. This implies that the canonical morphism $F_{|Z|}^{\nabla\infty} \rightarrow E_{|Z|}^{\nabla\infty}$ is an isomorphism. From the commutativity of the diagram (3.3.1.1), this isomorphism commutes with Frobenius. Moreover, from the unipotence property, we get that the canonical morphism $\text{can}: \mathcal{O}_K^{\text{an}} \otimes_K F_{|Z|}^{\nabla\infty} \rightarrow F_{|Z|}$ is an isomorphism. Then we notice that the composition of the horizontal arrows $F_{|Z|}^{\nabla\infty} \rightarrow F_{|Z|}/t_1 F_{|Z|}$ of the diagram (3.3.1.1) is an isomorphism, and from the commutativity of the diagram (3.3.1.1), this isomorphism commutes with Frobenius. On the other hand, by applying the functor i_1^* to $F \rightarrow F_{|Z|}$, we get an isomorphism commuting with Frobenius. Summing up, the canonical isomorphism $E_{|Z|}^{\nabla\infty} \xrightarrow{\sim} F/t_1 F$ commutes with Frobenius and with the monodromy operation induced by $t_1 \partial_1$.

Theorem 3.3.3 (Crew). *With the notation of §3.1, assume that \mathcal{E} is ι -pure of weight w and that \mathcal{E} comes from an overconvergent F -isocrystals on Y . Then $\operatorname{gr}_i^M(\mathcal{H}, N_{1,\mathcal{F}})|_{Z_1 \setminus D_1}$ (cf. paragraph 1.1.8 (iii)) is ι -pure of weight $w + 1 + i$ on the frame $(Z_1 \setminus D_1, Z_1, \mathcal{X})$.*

Proof. Let x be a closed point of $Z_1 \setminus \mathfrak{D}_1$, $k(x)$ its residue field and \mathcal{V}_x a finite étale \mathcal{V} -algebra which is a lifting of $k(x)$. We denote by $i_x: \operatorname{Spf} \mathcal{V}_x \hookrightarrow Z_1$ a lifting of the induced closed immersion by x . We denote $\operatorname{Spf} \mathcal{V}_x$ by $\{x\}$. We have to check that $i_x^! \operatorname{gr}_i^M(\mathcal{H}, N_{1,\mathcal{F}})(\dagger D_1)$ is ι -pure of weight $w + 1 + i$. Up to a change of basis of \mathcal{X} by the extension $\mathcal{V} \rightarrow \mathcal{V}_x$, we can suppose $k = k(x)$ by Lemma 2.2.10. Moreover, we can suppose that $\mathcal{Z} = Z_1$, and thus we suppose \mathfrak{D}_1 is empty and $\mathcal{Z} \cap V(t_2 - t_2(x), \dots, t_d - t_d(x)) = \{x\}$.

Let $\mathcal{X}_1 := \mathrm{Spf} A_1 := V(t_2 - t_2(x), \dots, t_d - t_d(x))$, $\mathcal{Y}_1 := \mathcal{X}_1 \setminus \mathcal{Z}$, $\alpha: \mathcal{X}_1 \hookrightarrow \mathcal{X}$, $\alpha^\#: (\mathcal{X}_1, \{x\}) \hookrightarrow (\mathcal{X}, \mathcal{Z})$, $\mathcal{F}_1 := \alpha^{\#*}(\mathcal{F})$, $\mathcal{E} := \mathcal{F}(\dagger Z)$. Since $t_1 \partial_1$ commutes with $\alpha^\#$, we reduce to the curve case, i.e. $\mathcal{X}_1 = \mathcal{X}$. Then, by using paragraph 3.3.2 and the remarks of 2.1.4 and 3.2.5, we apply [Cre98, Theorem 10.8]. \blacksquare

Remark 3.3.4. This theorem implies that $\ker(N_{1,\mathcal{F}})(-1) \parallel_{Z_1 \setminus D_1}$ (resp. $\mathrm{coker}(N_{1,\mathcal{F}}) \parallel_{Z_1 \setminus D_1}$) is ι -mixed of weight $\leq w + 1$ (resp. ι -mixed of weight $\geq w + 1$) by Remark 3.2.5.

3.4 Relations between a convergent log-isocrystal and its associated overconvergent isocrystal

In this subsection, we keep notation of §3.1 and we denote by $j := (\star, \mathrm{id}, \mathrm{id}): (Y, X, \mathcal{X}) \rightarrow (X, X, \mathcal{X})$ the canonical morphism of frames.

3.4.1. Since \mathcal{F} has nilpotent residues, from [CT12, 2.2.9], we get $u_+(\mathcal{F}) \xrightarrow{\sim} (\dagger Z)(\mathcal{F})$, and the same for $\mathbb{D}_{\mathcal{X}^\flat}(\mathcal{F})$. Hence, we have the isomorphisms

$$u_+ \circ \mathbb{D}_{\mathcal{X}^\flat}(\mathcal{F}) \xrightarrow{\sim} (\dagger Z) \circ \mathbb{D}_{\mathcal{X}^\flat}(\mathcal{F}) \xrightarrow{\sim} \mathbb{D}_{\mathcal{X},Z} \circ (\dagger Z)(\mathcal{F}) \xrightarrow{\sim} \mathbb{D}_{\mathcal{X},Z} \circ u_+(\mathcal{F}), \quad (3.4.1.1)$$

which are the identity over \mathcal{Y} . From [Car09a, 5.24.(ii)], we have the isomorphism

$$u_!(\mathcal{F}) \xrightarrow{\sim} u_+(\mathcal{F}(-Z)), \quad (3.4.1.2)$$

which is the biduality isomorphism over \mathcal{Y} . Then we get the isomorphism

$$j_!(\mathcal{E}) = \mathbb{D} \circ \mathbb{D}_{\mathcal{X},Z} \circ u_+(\mathcal{F}) \xrightarrow[(3.4.1.1)]{\sim} \mathbb{D} \circ u_+ \circ \mathbb{D}_{\mathcal{X}^\flat}(\mathcal{F}) = u_!(\mathcal{F}) \xrightarrow[(3.4.1.2)]{\sim} u_+(\mathcal{F}(-Z)), \quad (3.4.1.3)$$

which is the biduality isomorphism over \mathcal{Y} .

3.4.2. Let us introduce some notation. We denote by $\alpha_{1,\mathcal{F}}: \mathcal{F}(-Z_1) \hookrightarrow \mathcal{F}$ the canonical inclusion. Similarly to [Car09a, 5.24], we can check that the functor $\mathcal{H}^i u_{1+}$ ($i \neq 0$) vanishes for convergent isocrystals on \mathcal{X}^\flat . Thus, $\alpha_{1,\mathcal{F}}$ induces a homomorphism of coherent $\mathcal{D}_{\mathcal{X}^\#, \mathbb{Q}}^\dagger$ -modules $\beta_{1,\mathcal{F}} := u_{1+}(\alpha_{1,\mathcal{F}}): u_{1+}(\mathcal{F}(-Z_1)) \rightarrow u_{1+}(\mathcal{F})$.

Remark. Using the notation of §3.1 and paragraph 3.4.2, the kernel and cokernel of homomorphisms $\alpha_{1,\mathcal{F}}$ and $\beta_{1,\mathcal{F}}$ are supported in Z_1 . Then, $f^{\#!} \ker(\beta_{1,\mathcal{F}})$ and $f^{\#!} \mathrm{coker}(\beta_{1,\mathcal{F}})$ are coherent $\mathcal{D}_{\mathcal{X}^\#, \mathbb{Q}}^\dagger$ -modules with support in Z_1 . Since $g_1^\# \circ i_{1+}^{\dagger\#} = \mathrm{id}$, by the logarithmic version of Berthelot-Kashiwara's theorem (cf. [Car12c, 5.3.6]), we have

$$\begin{aligned} g_{1+}^\#(f^{\#!} \ker(\beta_{1,\mathcal{F}})) &\xrightarrow{\sim} i_1^{\dagger\#!} f^{\#!} \ker(\beta_{1,\mathcal{F}}) \xrightarrow{\sim} i_1^{\#!} \ker(\beta_{1,\mathcal{F}}), \\ g_{1+}^\#(f^{\#!} \mathrm{coker}(\beta_{1,\mathcal{F}})) &\xrightarrow{\sim} i_1^{\dagger\#!} f^{\#!} \mathrm{coker}(\beta_{1,\mathcal{F}}) \xrightarrow{\sim} i_1^{\#!} \mathrm{coker}(\beta_{1,\mathcal{F}}). \end{aligned} \quad (3.4.2.1)$$

Moreover, they are convergent isocrystals on $Z_1^\#$ by [Car12b, 3.5.5].

Lemma 3.4.3. *The canonical morphism $u_{1+}(\mathcal{F}) \rightarrow (\dagger Z_1) \circ u_{1+}(\mathcal{F})$ is an isomorphism, and by [CT12, 2.2.1], we have an isomorphism $\rho_{1,\mathcal{F}}: u_{1+}(\mathcal{F}) \xrightarrow{\sim} (\dagger Z_1)(\mathcal{F})$.*

Proof. Since \mathcal{F} has nilpotent residues, this follows from [Car12b, 3.5.6.2]. \blacksquare

Lemma 3.4.4. *We have the canonical isomorphism $\iota_{1,\mathcal{F}}: u_{1!}(\mathcal{F}) \xrightarrow{\sim} u_{1+}(\mathcal{F}(-Z_1))$, which is the biduality isomorphism outside Z_1 .*

Proof. The proof is the same as that of [Car09a, 5.24.(ii)]. \blacksquare

Lemma 3.4.5. Let \mathcal{P} be a smooth formal scheme, \mathcal{T} be a strict normal crossing divisor on \mathcal{P} , $\mathcal{P}^\# := (\mathcal{P}, \mathcal{T})$, H be a divisor of \mathcal{P} . Consider the following diagram of coherent $\mathcal{D}_{\mathcal{P}^\#, \mathbb{Q}}^\dagger$ -modules:

$$\begin{array}{ccc} \mathcal{M}_1 & \longrightarrow & \mathcal{M}'_1 \\ \downarrow & & \downarrow \\ \mathcal{M}_2 & \longrightarrow & \mathcal{M}'_2. \end{array}$$

We suppose that the canonical morphism $\mathcal{M}'_2 \rightarrow \mathcal{M}'_2(\dagger H) := \mathcal{D}_{\mathcal{P}^\#}^\dagger(\dagger H)_{\mathbb{Q}} \otimes_{\mathcal{D}_{\mathcal{P}^\#, \mathbb{Q}}^\dagger} \mathcal{M}$ is an isomorphism. Then the diagram is commutative if and only if it is so after restricting to $\mathcal{P} \setminus H$.

Proof. Use [Car09a, 4.7]. ■

Proposition 3.4.6. (i) The following diagram of coherent $\mathcal{D}_{\mathcal{X}^\#, \mathbb{Q}}^\dagger$ -modules

$$\begin{array}{ccccc} \mathbb{D}_{\mathcal{X}^\#} u_{1+} \mathbb{D}_{\mathcal{X}^\flat}(\mathcal{F}) & \xrightarrow[\sim]{\iota_{1, \mathcal{F}}} & u_{1+}(\mathcal{F}(-Z_1)) & & \\ \mathbb{D}_{\mathcal{X}^\#}(\beta_{1, \mathbb{D}_{\mathcal{X}^\flat}(\mathcal{F})}) \downarrow & & \downarrow \beta_{1, \mathcal{F}} & & \\ \mathbb{D}_{\mathcal{X}^\#} \circ u_{1+} \circ (-Z_1) \circ \mathbb{D}_{\mathcal{X}^\flat}(\mathcal{F}) & \xrightarrow[\mathbb{D}_{\mathcal{X}^\#}(\iota_{1, \mathbb{D}_{\mathcal{X}^\flat}(\mathcal{F})})]{\sim} & \mathbb{D}_{\mathcal{X}^\#} \circ u_{1!} \circ \mathbb{D}_{\mathcal{X}^\flat}(\mathcal{F}) & \xrightarrow[\star]{\sim} & u_{1+}(\mathcal{F}) \end{array}$$

where \star is induced by functoriality using the biduality isomorphisms $\mathbb{D}_{\mathcal{X}^\flat} \circ \mathbb{D}_{\mathcal{X}^\flat} \xrightarrow{\sim} \text{id}$ and $\mathbb{D}_{\mathcal{X}^\#} \circ \mathbb{D}_{\mathcal{X}^\#} \xrightarrow{\sim} \text{id}$, is commutative.

(ii) We have the isomorphisms of coherent $\mathcal{D}_{\mathcal{X}^\#, \mathbb{Q}}^\dagger$ -modules with support in Z_1 :

$$\ker(\beta_{1, \mathcal{F}}) \xrightarrow{\sim} \mathbb{D}_{\mathcal{X}^\#}(\text{coker}(\beta_{1, \mathbb{D}_{\mathcal{X}^\flat}(\mathcal{F})})), \quad \text{coker}(\beta_{1, \mathcal{F}}) \xrightarrow{\sim} \mathbb{D}_{\mathcal{X}^\#}(\ker(\beta_{1, \mathbb{D}_{\mathcal{X}^\flat}(\mathcal{F})})). \quad (3.4.6.1)$$

Proof. Let us show (i). By Lemma 3.4.3, we have the isomorphism $u_{1+}(\mathcal{F}) \xrightarrow{\sim} \mathcal{F}(\dagger Z_1)$. Since the diagram is commutative outside of Z_1 , the commutativity follows by Lemma 3.4.5. The second part of the proposition is straightforward from the first one. ■

Lemma 3.4.7. We have the canonical isomorphisms of convergent isocrystals on $Z_1^\#$ with nilpotent residues:

$$i_1^{\#1}(\ker(\beta_{1, \mathcal{F}})) \xrightarrow{\sim} \ker(N_{1, \mathcal{F}}), \quad i_1^{\#1}(\text{coker}(\beta_{1, \mathcal{F}})) \xrightarrow{\sim} \text{coker}(N_{1, \mathcal{F}}). \quad (3.4.7.1)$$

Proof. By Remark 3.4.2, we already know that $i_1^{\#1}(\ker(\beta_{1, \mathcal{F}}))$ and $i_1^{\#1}(\text{coker}(\beta_{1, \mathcal{F}}))$ are convergent isocrystals on $Z_1^\#$ (with nilpotent residues). From Kedlaya's full-faithfulness theorem [Ked07, 6.4.5], we reduce to the case where \mathfrak{D} is empty, and we can remove $\#$ everywhere in the notation. Since the left square of (3.1.2.1) is cartesian, the functor $f^!$ is an equivalence of categories between the category of coherent $\mathcal{D}_{\mathcal{X}, \mathbb{Q}}^\dagger$ -modules with support in Z_1 and the category of coherent $\mathcal{D}_{\mathcal{X}', \mathbb{Q}}^\dagger$ -modules with support in Z_1 , by Berthelot-Kashiwara's theorem. By [Car12c, 5.4.6], we have $f^!(\beta_{1, \mathcal{F}}) \xrightarrow{\sim} \beta_{1, f^\flat(\mathcal{F})}$ and $f^!(N_{1, \mathcal{F}}) \xrightarrow{\sim} N_{1, f^\flat(\mathcal{F})}$. So, we are reduced to the case where $f = \text{id}$.

Now, from the relative duality isomorphism, for any coherent $\mathcal{D}_{\mathcal{X}, \mathbb{Q}}^\dagger$ -module \mathcal{G} with support in Z_1 , we get the isomorphism $\mathbb{D}_{Z_1} i_1^! \mathcal{G} \cong i_1^! \mathbb{D}_{\mathcal{X}} \mathcal{G}$. Thus, by duality, Lemma 3.2.4, and (3.4.6.1), it remains to check the second isomorphism. We get

$$g_{1+}(\text{coker}(\beta_{1, \mathcal{F}})) \xrightarrow{\sim} \mathcal{H}^0 g_{1+}(\text{coker}(\beta_{1, \mathcal{F}})) \xrightarrow{\sim} \text{coker}(\mathcal{H}^0 g_{1+}(\beta_{1, \mathcal{F}})),$$

where the first isomorphism follows by Remark 3.4.2 and the second by the right exactness of $\mathcal{H}^0 g_{1+}$. Arguing similarly to [Car09a, 5.24], the functor $\mathcal{H}^i u_{1+}$ ($i \neq 0$) vanishes for convergent isocrystals on \mathcal{X}^\flat . Since $g_1 \circ u_1 = g_1^\flat$, we get

$$(\mathcal{H}^0 g_{1+})(\beta_{1, \mathcal{F}}) = (\mathcal{H}^0 g_{1+}) \circ u_{1+}(\alpha_{1, \mathcal{F}}) \xrightarrow{\sim} \mathcal{H}^0 g_{1+}^\flat(\alpha_{1, \mathcal{F}}).$$

Hence $g_{1+}(\operatorname{coker}(\beta_{1,\mathcal{F}})) \xrightarrow{\sim} \operatorname{coker}(\mathcal{H}^0 g_{1+}^b(\alpha_{1,\mathcal{F}})) \xrightarrow{\sim} \mathcal{H}^0 g_{1+}^b \operatorname{coker}(\alpha_{1,\mathcal{F}})$. Since $\operatorname{coker}(\alpha_{1,\mathcal{F}}) = \mathcal{F}/\mathcal{F}(-\mathcal{Z}_1)$ and $\mathcal{H}^0 g_{1+}^b \xrightarrow{\sim} \operatorname{coker} \iota_{1,\partial_1}$, we get

$$g_{1+}(\operatorname{coker}(\beta_{1,\mathcal{F}})) \xrightarrow{\sim} \operatorname{coker}(N_{1,\mathcal{F}}).$$

With the isomorphism of (3.4.2.1), the lemma follows. \blacksquare

3.4.8. Let $\epsilon_{1,\mathcal{F}} := u_+ \circ (-\mathfrak{D})(\alpha_{1,\mathcal{F}}): u_+(\mathcal{F}(-\mathcal{Z})) \rightarrow u_+(\mathcal{F}(-\mathfrak{D}))$ and $\theta_{1,\mathcal{E}}: \mathbb{D} \circ \mathbb{D}_Z(\mathcal{E}) \rightarrow (\dagger Z_1) \circ \mathbb{D} \circ \mathbb{D}_Z(\mathcal{E}) \xrightarrow{\sim} \mathbb{D}_{Z_1} \circ \mathbb{D}_Z(\mathcal{E})$ be the canonical homomorphisms. Both homomorphisms are isomorphisms outside Z_1 .

Remark. (i) Since the functor $(-\mathfrak{D})$ commutes canonically with u_{1+} , we get $u_{1+} \circ (-\mathfrak{D})(\alpha_{1,\mathcal{F}}) \xrightarrow{\sim} (-\mathfrak{D})(\beta_{1,\mathcal{F}})$. By applying v_+ to this isomorphism we get

$$\epsilon_{1,\mathcal{F}} \xrightarrow{\sim} v_+ \circ u_{1+} \circ (-\mathfrak{D})(\alpha_{1,\mathcal{F}}) \xrightarrow{\sim} v_+ \circ (-\mathfrak{D})(\beta_{1,\mathcal{F}}).$$

(ii) Since the functor $(-\mathfrak{D})$ commutes canonically with u_{1+} and with $(\dagger Z_1)$, by applying $(-\mathfrak{D})$ to $\rho_{1,\mathcal{F}}$ (cf. Lemma 3.4.3), we get $(\dagger Z_1) \circ (-\mathcal{Z})(\mathcal{F}) \xrightarrow{\sim} (\dagger Z_1) \circ (-\mathfrak{D})(\mathcal{F}) \xleftarrow{\sim} u_{1+} \circ (-\mathfrak{D})(\mathcal{F})$. Applying v_+ , we obtain

$$(\dagger Z_1) \circ u_+ \circ (-\mathcal{Z})(\mathcal{F}) \xrightarrow{\sim} v_+ \circ (\dagger Z_1) \circ (-\mathcal{Z})(\mathcal{F}) \xrightarrow{\sim} v_+ \circ u_{1+} \circ (-\mathfrak{D})(\mathcal{F}) \xrightarrow{\sim} u_+ \circ (-\mathfrak{D})(\mathcal{F}),$$

where the first isomorphism is induced by $(\dagger Z_1) \circ u_+ \xrightarrow{\sim} v_+ \circ (\dagger Z_1)$ from [CT12, 2.2.2]. This homomorphism is the identity outside Z_1 .

(iii) By applying $(\dagger Z_1)$ to (3.4.1.3), we obtain

$$\mathbb{D}_{Z_1} \circ \mathbb{D}_Z \circ u_+(\mathcal{F}) \xrightarrow{\sim} (\dagger Z_1)(u_+(\mathcal{F}(-\mathcal{Z}))) \xrightarrow[\text{(ii)}]{\sim} u_+(\mathcal{F}(-\mathfrak{D})). \quad (3.4.8.1)$$

Proposition 3.4.9. (i) The following diagram of canonical homomorphisms is commutative:

$$\begin{array}{ccc} \mathbb{D} \circ \mathbb{D}_Z(\mathcal{E}) & \xrightarrow{\theta_{1,\mathcal{E}}} & \mathbb{D}_{Z_1} \circ \mathbb{D}_Z(\mathcal{E}) \\ (3.4.1.3) \downarrow \sim & & \sim \downarrow (3.4.8.1) \\ u_+(\mathcal{F}(-\mathcal{Z})) & \xrightarrow{\epsilon_{1,\mathcal{F}}} & u_+(\mathcal{F}(-\mathfrak{D})). \end{array} \quad (3.4.9.1)$$

(ii) We have the canonical isomorphisms:

$$\mathcal{H}_{Z_1}^{\dagger 0}(j_!(\mathcal{E})) \xrightarrow{\sim} \ker(\theta_{1,\mathcal{E}}) \xrightarrow{\sim} \ker(\epsilon_{1,\mathcal{F}}), \quad \mathcal{H}_{Z_1}^{\dagger 1}(j_!(\mathcal{E})) \xrightarrow{\sim} \operatorname{coker}(\theta_{1,\mathcal{E}}) \xrightarrow{\sim} \operatorname{coker}(\epsilon_{1,\mathcal{F}}). \quad (3.4.9.2)$$

Proof. Thanks to Lemma 3.4.5, it is sufficient to check the commutativity of the square (3.4.9.1) outside Z_1 , which is easy. For the second, the first isomorphism follows from the localization triangle, and the second by (i). \blacksquare

Proposition 3.4.10. We have the canonical isomorphisms:

$$i_1^! \ker(\epsilon_{1,\mathcal{F}}) \xrightarrow{\sim} w_{1+}(\ker(N_{1,\mathcal{F}})(-\mathfrak{D}_1)), \quad i_1^! \operatorname{coker}(\epsilon_{1,\mathcal{F}}) \xrightarrow{\sim} w_{1+}(\operatorname{coker}(N_{1,\mathcal{F}})(-\mathfrak{D}_1)). \quad (3.4.10.1)$$

Proof. By Remark 3.4.8, we have $u_{1+} \circ (-\mathfrak{D})(\alpha_{1,\mathcal{F}}) \xrightarrow{\sim} (-\mathfrak{D})(\beta_{1,\mathcal{F}})$. Thus, we get the exact sequence

$$0 \rightarrow \ker(\beta_{1,\mathcal{F}})(-\mathfrak{D}) \rightarrow u_{1+}(\mathcal{F}(-\mathcal{Z})) \xrightarrow{u_{1+} \circ (-\mathfrak{D})(\alpha_{1,\mathcal{F}})} u_{1+}(\mathcal{F}(-\mathfrak{D})) \rightarrow \operatorname{coker}(\beta_{1,\mathcal{F}})(-\mathfrak{D}) \rightarrow 0. \quad (\star)$$

We have $(-\mathfrak{D}_1) \circ i_1^{\#1} \xrightarrow{\sim} i_1^{\#1} \circ (-\mathfrak{D})$. Then, by the logarithmic version of Berthelot-Kashiwara's theorem [Car12c, 5.3.6],

$$\ker(\beta_{1,\mathcal{F}})(-\mathfrak{D}) \xleftarrow{\sim} i_{1+}^{\#} \circ i_1^{\#1}(\ker(\beta_{1,\mathcal{F}})(-\mathfrak{D})) \xrightarrow{\sim} i_{1+}^{\#}((-\mathfrak{D}_1) \circ i_1^{\#1}(\ker(\beta_{1,\mathcal{F}}))). \quad (\star\star)$$

By Lemma 3.4.7, $(-\mathfrak{D}_1) \circ i_1^{\#1}(\ker(\beta_{1,\mathcal{F}}))$ is a convergent isocrystals on $\mathcal{Z}_1^{\#}$. Arguing similarly to [Car09a, 5.24], the functor $\mathcal{H}^0 w_{1+}$ is acyclic (i.e. $\mathcal{H}^i w_{1+}$ vanishes for $i \neq 0$) for convergent isocrystals on $\mathcal{Z}_1^{\#}$. Since the

functor $i_{1+}^\#$ is exact, these imply that $\ker(\beta_{1,\mathcal{F}})(-\mathfrak{D})$ is acyclic for $\mathcal{H}^0 v_+$. In the same way, $\operatorname{coker}(\beta_{1,\mathcal{F}})(-\mathfrak{D})$ is acyclic for $\mathcal{H}^0 v_+$. Since $u_{1+}(\mathcal{F}(-\mathcal{Z}))$ and $u_{1+}(\mathcal{F}(-\mathfrak{D}))$ are convergent isocrystals on $\mathcal{X}^\#$, similarly to [Car09a, 5.24], $\mathcal{H}^0 v_+$ is acyclic for these modules. Thus, by applying the functor v_+ to (\star) , we obtain the exact sequence:

$$0 \rightarrow v_+(\ker(\beta_{1,\mathcal{F}})(-\mathfrak{D})) \rightarrow u_+(\mathcal{F}(-\mathcal{Z})) \xrightarrow{\epsilon_{1,\mathcal{F}}} u_+(\mathcal{F}(-\mathfrak{D})) \rightarrow v_+(\operatorname{coker}(\beta_{1,\mathcal{F}})(-\mathfrak{D})) \rightarrow 0.$$

On the other hand, by applying the exact functor $i_{1+} \circ w_{1+} \circ (-\mathfrak{D}_1)$ to (3.4.7.1), we get

$$v_+(\ker(\beta_{1,\mathcal{F}})(-\mathfrak{D})) \xrightarrow[\sim]{v_+(\star\star)} i_{1+} \circ w_{1+} \circ (-\mathfrak{D}_1) \circ i_1^\#(\ker(\beta_{1,\mathcal{F}})) \xrightarrow{\sim} i_{1+} \circ w_{1+} \circ (-\mathfrak{D}_1)(\ker(N_{1,\mathcal{F}})).$$

If we replace \ker by coker , we get the same isomorphism, and we conclude the proof. \blacksquare

Remark 3.4.11. The homomorphisms $\epsilon_{1,\mathcal{F}}$ and $\beta_{1,\mathcal{F}}$ are identity outside of D , and the isomorphisms in Lemma 3.4.7 and Lemma 3.4.10 are identity outside of D_1 .

3.4.12. We put $\mathcal{Y}_1 := \mathcal{Z}_1 \setminus \mathfrak{D}_1$, and let $j_1 := (\star, \operatorname{id}, \operatorname{id}): (Y_1, Z_1, \mathcal{Z}_1) \rightarrow (Z_1, Z_1, \mathcal{Z}_1)$ be the canonical morphism of frames. We get the isomorphisms

$$\begin{aligned} i_1^! \mathcal{H}_{Z_1}^{\dagger 1}(j_1(\mathcal{E})) &\xrightarrow{\sim} i_1^!(\operatorname{coker}(\theta_{1,\mathcal{E}})) \xrightarrow{\sim} i_1^!(\operatorname{coker}(\epsilon_{1,\mathcal{F}})) \xrightarrow{\sim} w_{1+}(\operatorname{coker} N_{1,\mathcal{F}}(-\mathfrak{D}_1)) \\ &\xrightarrow{\sim} j_{1!} \circ w_{1+}(\operatorname{coker} N_{1,\mathcal{F}}) \xrightarrow{\sim} j_{1!} \circ (\dagger D_1)(\operatorname{coker} N_{1,\mathcal{F}}), \end{aligned}$$

where the second isomorphism follows by (3.4.9.2), the third by (3.4.10.1), the fourth is similarly to (3.4.1.3), and the last one by [Car12b, 3.5.6.2]. We proceed similarly for kernel. Summing up, we get the isomorphisms:

$$i_1^! \mathcal{H}_{Z_1}^{\dagger 1}(j_1(\mathcal{E})) \xrightarrow{\sim} j_{1!} \circ (\dagger D_1)(\operatorname{coker} N_{1,\mathcal{F}}), \quad i_1^! \mathcal{H}_{Z_1}^{\dagger 0}(j_1(\mathcal{E})) \xrightarrow{\sim} j_{1!} \circ (\dagger D_1)(\ker N_{1,\mathcal{F}}). \quad (3.4.12.1)$$

Lemma 3.4.13. Let (Y, X, \mathcal{P}) be a frame, and $j = (\star, \operatorname{id}, \operatorname{id}): (Y, X, \mathcal{P}) \rightarrow (X, X, \mathcal{P})$ be a morphism of frames. Let $\mathcal{G}_1 \in F\text{-Ovhol}(X, \mathcal{P}/K)$, $\mathcal{G}_2 \in F\text{-Ovhol}(Y, \mathcal{P}/K)$, and $\phi: \mathcal{G}_1 \xrightarrow{\sim} j_!(\mathcal{G}_2)$ be an isomorphism in $\text{Ovhol}(X, \mathcal{P}/K)$. Let \mathcal{U} be an open formal subscheme of \mathcal{P} containing Y . If $\phi|_{\mathcal{U}}$ commutes with the Frobenius structures, then so is ϕ .

Proof. Let us show that ϕ^{-1} is compatible with Frobenius. By taking the adjoint, it suffices to show that the homomorphism $j^!(\phi^{-1}): \mathcal{G}_2 \rightarrow \mathcal{G}_1$ in $\text{Ovhol}(Y, \mathcal{P})$ is compatible with Frobenius. The lemma follows since the restriction functor $\text{Ovhol}(Y, \mathcal{P}) \rightarrow \text{Ovhol}(Y, \mathcal{U})$ is faithful by Remark 1.2.9. \blacksquare

3.5 An isomorphism compatible with Frobenius

3.5.1. In this subsection, we keep the notation and hypotheses of §3.1, and we assume that $f = \operatorname{id}$ and that D_1 is empty, unless otherwise stated. We suppose also that there exists an étale morphism $h: \mathcal{X} \rightarrow \widehat{\mathbb{A}}_{Z_1}^1$ such that g_1 is the composition of h and the projection $\widehat{\mathbb{A}}_{Z_1}^1 \rightarrow Z_1$, and that the composition of i_1 and h is the zero section. We define $F_{\mathcal{X}/Z_1}$ so that we get the commutativity of the canonical diagram on the left whose squares of the bottom are cartesian:

$$\begin{array}{ccc} Z_1 \hookrightarrow \mathcal{X} & \xrightarrow{g_1} & Z_1 \\ \parallel & \downarrow F_{\mathcal{X}/Z_1} & \parallel \\ Z_1 \hookrightarrow \mathcal{X}' & \xrightarrow{g_1} & Z_1 \\ F_{Z_1} \downarrow & \square & \downarrow F_{Z_1} \\ Z_1 \hookrightarrow \mathcal{X} & \xrightarrow{g_1} & Z_1, \end{array} \quad \begin{array}{ccc} \mathcal{X}' & \longrightarrow & \widehat{\mathbb{A}}_{Z_1}^1 \longrightarrow \widehat{\mathbb{A}}_{\mathcal{V}}^n \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{h} & \widehat{\mathbb{A}}_{Z_1}^1 \longrightarrow \widehat{\mathbb{A}}_{\mathcal{V}}^n. \end{array} \quad (3.5.1.1)$$

By abuse of notation, we also denote by F_{Z_1} the morphisms $\mathcal{X}' \rightarrow \mathcal{X}$ and $\widehat{\mathbb{A}}_{Z_1}^1 \rightarrow \widehat{\mathbb{A}}_{Z_1}^1$ induced by F_{Z_1} . We denote by t'_1, \dots, t'_d the corresponding local coordinate of \mathcal{X}' . Hence, t_1 (resp. t'_1) is a relative local coordinate of \mathcal{X}/Z_1 (resp. \mathcal{X}'/Z_1).

3.5.2. We put $\mathcal{E}' := F_{\mathcal{Z}_1}^*(\mathcal{E})$, $\mathcal{F}' := F_{\mathcal{Z}_1}^*(\mathcal{F})$. We get the commutative diagram:

$$\begin{array}{ccccc} F_{\mathcal{X}/\mathcal{Z}_1}^* \mathcal{E}' & \xrightarrow{\sim} & F_{\mathcal{X}}^*(\mathcal{E}) & \xrightarrow[\phi_{\mathcal{E}}]{\sim} & \mathcal{E} \\ \uparrow & & \uparrow & & \uparrow \\ F_{\mathcal{X}/\mathcal{Z}_1}^* \mathcal{F}' & \xrightarrow{\sim} & F_{\mathcal{X}}^*(\mathcal{F}) & \xrightarrow[\phi_{\mathcal{F}}]{\sim} & \mathcal{F} \end{array} \quad (3.5.2.1)$$

3.5.3. By [Abe13, 3.14], we have the canonical morphism compatible with Frobenius:

$$g_{1+}(\mathcal{E}) \xrightarrow{\sim} g_{1*}(\Omega_{\mathcal{X}/\mathcal{Z}_1}^\bullet \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{E})1. \quad (3.5.3.1)$$

By identifying $\Omega_{\mathcal{X}/\mathcal{Z}_1}^1$ with $\mathcal{O}_{\mathcal{X}}$ and applying the functor \mathcal{H}^0 , we get the isomorphism

$$\mathcal{H}^0 g_{1+}(\mathcal{E}) \xrightarrow{\sim} \text{coker}_{\partial_1}(\mathcal{E})(1). \quad (3.5.3.2)$$

Let us recall the Frobenius structure on $\text{coker}_{\partial_1}(\mathcal{E})$. We denote by $\text{adj}: \mathcal{E}' \rightarrow F_{\mathcal{X}/\mathcal{Z}_1}^* \mathcal{E}' \xrightarrow{\sim} F_{\mathcal{X}}^* \mathcal{E}$ the adjunction homomorphism. By identifying $\Omega_{\mathcal{X}'/\mathcal{Z}_1}^1$ with $\mathcal{O}_{\mathcal{X}'}$ and $\Omega_{\mathcal{X}/\mathcal{Z}_1}^1$ with $\mathcal{O}_{\mathcal{X}}$, the canonical quasi-isomorphism $\Omega_{\mathcal{X}'/\mathcal{Z}_1}^\bullet \otimes_{\mathcal{O}_{\mathcal{X}'}} \mathcal{E}' \rightarrow \Omega_{\mathcal{X}/\mathcal{Z}_1}^\bullet \otimes_{\mathcal{O}_{\mathcal{X}}} F^*(\mathcal{E})$ is given by the commutative diagram

$$\begin{array}{ccc} \mathcal{E}' & \xrightarrow{\text{adj}} & F_{\mathcal{X}}^* \mathcal{E} \\ \downarrow \partial'_1 & & \downarrow \partial_1 \\ \mathcal{E}' & \xrightarrow{qt^{q-1} \cdot \text{adj}} & F_{\mathcal{X}}^* \mathcal{E} \end{array} \quad (3.5.3.3)$$

The maps $qt_1^{q-1} \cdot \text{adj}$ induces the isomorphism $\text{coker}_{\partial'_1}(\mathcal{E}') \xrightarrow{\sim} \text{coker}_{\partial_1}(F_{\mathcal{X}}^* \mathcal{E})$ given by $[x'] \mapsto [qt_1^{q-1} \otimes x']$. Hence we get

$$F_{\mathcal{Z}_1}^* \text{coker}_{\partial_1}(\mathcal{E}) \xrightarrow{\sim} \text{coker}_{\partial'_1}(\mathcal{E}') \xrightarrow{\sim} \text{coker}_{\partial_1}(F_{\mathcal{X}}^* \mathcal{E}) \xrightarrow[\phi]{\sim} \text{coker}_{\partial_1}(\mathcal{E}), \quad (3.5.3.4)$$

where the first homomorphism is an isomorphism since $F_{\mathcal{Z}_1}$ is flat.

3.5.4. We have the canonical morphism (without Frobenius structure):

$$g_{1+}^b(\mathcal{F}) \xrightarrow{\sim} g_{1*}(\Omega_{\mathcal{X}^b/\mathcal{Z}_1}^\bullet \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F})[1]. \quad (3.5.4.1)$$

By identifying $\Omega_{\mathcal{X}^b/\mathcal{Z}_1}$ with $\mathcal{O}_{\mathcal{X}}$ and by applying the functor \mathcal{H}^0 , we get the isomorphism

$$\mathcal{H}^0 g_{1+}^b(\mathcal{F}) \xrightarrow{\sim} \text{coker}_{t_1 \partial_1}(\mathcal{F}). \quad (3.5.4.2)$$

Now, let us define a Frobenius structure on $\text{coker}_{t_1 \partial_1}(\mathcal{F})$ as follows: we denote by $\text{adj}: \mathcal{F}' \rightarrow F_{\mathcal{X}/\mathcal{Z}_1}^* \mathcal{F}' \xrightarrow{\sim} F_{\mathcal{X}}^* \mathcal{F}$ the canonical homomorphism. By (3.2.8.1), we can check that the diagram

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{\text{adj}} & F_{\mathcal{X}}^* \mathcal{F} \\ \downarrow t'_1 \partial'_1 & & \downarrow t_1 \partial_1 \\ \mathcal{F}' & \xrightarrow{q \cdot \text{adj}} & F_{\mathcal{X}}^* \mathcal{F} \end{array} \quad (3.5.4.3)$$

is commutative. Thus the map $q \cdot \text{adj}$ induces the canonical map $\text{coker}_{t'_1 \partial'_1}(\mathcal{F}') \rightarrow \text{coker}_{t_1 \partial_1}(F_{\mathcal{X}}^* \mathcal{F})$. We get the Frobenius structure on $\text{coker}_{t_1 \partial_1}(\mathcal{F})$ via the homomorphisms:

$$F_{\mathcal{Z}_1}^* \text{coker}_{t_1 \partial_1}(\mathcal{F}) \xrightarrow{\sim} \text{coker}_{t'_1 \partial'_1}(\mathcal{F}') \xrightarrow[\text{coker}_{t_1 \partial_1}(\phi)]{\sim} \text{coker}_{t_1 \partial_1}(F_{\mathcal{X}}^* \mathcal{F}) \xrightarrow{\sim} \text{coker}_{t_1 \partial_1}(\mathcal{F}). \quad (3.5.4.4)$$

3.5.5. We have the isomorphisms

$$g_{1*}(\Omega_{\mathcal{X}^b/\mathcal{Z}_1}^\bullet \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}[1]) \xleftarrow[\text{(3.5.4.1)}]{\sim} g_{1+}^b(\mathcal{F}) \xleftarrow{\sim} g_{1+}(u_+(\mathcal{F})) \xrightarrow{\sim} g_{1+}(\mathcal{E}) \xrightarrow[\text{(3.5.3.1)}]{\sim} g_{1*}(\Omega_{\mathcal{X}/\mathcal{Z}_1}^\bullet \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{E}[1]). \quad (3.5.5.1)$$

Lemma 3.5.6. *By applying \mathcal{H}^0 to the isomorphism (3.5.5.1), we obtain the isomorphism $\text{coker } t_1 \partial_1(\mathcal{F}) \xrightarrow{\sim} \text{coker } \partial_1(\mathcal{E})$. This isomorphism commutes with Frobenius.*

Proof. Let us recall the definition of the isomorphisms appearing in (3.5.5.1). Since $\mathcal{O}_{\mathcal{X}}(\mathcal{Z}_1) = \mathcal{H}\text{om}_{\mathcal{O}_{\mathcal{X}}}(\omega_{\mathcal{X}}, \omega_{\mathcal{X}^b})$, we have $\mathcal{D}_{\mathcal{X} \leftarrow \mathcal{X}^b}^\dagger = (\mathcal{D}_{\mathcal{X}}^\dagger \otimes_{\mathcal{O}_{\mathcal{X}}} \omega_{\mathcal{X}}^{-1}) \otimes_{\mathcal{O}_{\mathcal{X}}} \omega_{\mathcal{X}^b} \xrightarrow{\sim} \mathcal{D}_{\mathcal{X}}^\dagger \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(\mathcal{Z}_1) = \mathcal{D}_{\mathcal{X}}^\dagger(\mathcal{Z}_1)$. The evaluation isomorphism $\omega_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(\mathcal{Z}_1) \xrightarrow{\sim} \omega_{\mathcal{X}^b}$ induces the canonical isomorphism $\mathcal{D}_{\mathcal{Z}_1 \leftarrow \mathcal{X}}^\dagger \otimes_{\mathcal{D}_{\mathcal{X}}^\dagger}^{\mathbb{L}} \mathcal{D}_{\mathcal{X} \leftarrow \mathcal{X}^b}^\dagger \xrightarrow{\sim} \mathcal{D}_{\mathcal{Z}_1 \leftarrow \mathcal{X}^b}^\dagger$, which induces the isomorphism $g_{1+}^b \xleftarrow{\sim} g_{1+} \circ u_+$. The exact sequence

$$0 \rightarrow \mathcal{D}_{\mathcal{X}^{(b)}}^\dagger \rightarrow \Omega_{\mathcal{X}^{(b)}/\mathcal{Z}_1}^1 \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D}_{\mathcal{X}^{(b)}}^\dagger \rightarrow \mathcal{D}_{\mathcal{Z}_1 \leftarrow \mathcal{X}^{(b)}}^\dagger \rightarrow 0 \quad (\star)$$

induces the isomorphism (3.5.3.1) (resp. (3.5.4.1)).

Now, consider the following diagram

$$\begin{array}{ccccc} (\Omega_{\mathcal{X}/\mathcal{Z}_1}^1 \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D}_{\mathcal{X}}^\dagger) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(\mathcal{Z}_1) & \xrightarrow[\sim]{\tau} & \Omega_{\mathcal{X}/\mathcal{Z}_1}^1 \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(\mathcal{Z}_1) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D}_{\mathcal{X}}^\dagger & \longrightarrow & \Omega_{\mathcal{X}/\mathcal{Z}_1}^1 \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(\mathcal{Z}_1) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D}_{\mathcal{X} \rightarrow \mathcal{Z}_1}^\dagger \\ \uparrow & & \uparrow & & \uparrow \sim \\ \Omega_{\mathcal{X}^b/\mathcal{Z}_1}^1 \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D}_{\mathcal{X}^b}^\dagger & \xrightarrow[\tau^b]{\sim} & \Omega_{\mathcal{X}^b/\mathcal{Z}_1}^1 \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D}_{\mathcal{X}^b}^\dagger & \longrightarrow & \Omega_{\mathcal{X}^b/\mathcal{Z}_1}^1 \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D}_{\mathcal{X}^b \rightarrow \mathcal{Z}_1}^\dagger \end{array}$$

where the right (resp. middle) vertical isomorphism is given by the inverse of the evaluation homomorphism and the inclusion $\mathcal{D}_{\mathcal{X}^b}^\dagger \subset \mathcal{D}_{\mathcal{X}}^\dagger$, τ and τ^b are the transposition isomorphisms. The right square is commutative since $\mathcal{D}_{\mathcal{X}^b \rightarrow \mathcal{Z}_1}^\dagger \cong \mathcal{D}_{\mathcal{X} \rightarrow \mathcal{Z}_1}^\dagger$. Then, we get the factorization $\Omega_{\mathcal{X}^b/\mathcal{Z}_1}^1 \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D}_{\mathcal{X}^b}^\dagger \rightarrow (\Omega_{\mathcal{X}/\mathcal{Z}_1}^1 \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D}_{\mathcal{X}}^\dagger) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(\mathcal{Z}_1)$ so that the diagram is commutative. Then, we get the right square of the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{D}_{\mathcal{X}}^\dagger \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(\mathcal{Z}_1) & \xrightarrow{\nabla} & \Omega_{\mathcal{X}/\mathcal{Z}_1}^1 \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D}_{\mathcal{X}}^\dagger \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(\mathcal{Z}_1) & \longrightarrow & \mathcal{D}_{\mathcal{Z}_1 \leftarrow \mathcal{X}}^\dagger \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(\mathcal{Z}_1) \longrightarrow 0 \\ & & \uparrow \text{dotted} & & \uparrow & & \sim \uparrow \\ 0 & \longrightarrow & \mathcal{D}_{\mathcal{X}^b}^\dagger & \xrightarrow{\nabla^b} & \Omega_{\mathcal{X}^b/\mathcal{Z}_1}^1 \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D}_{\mathcal{X}^b}^\dagger & \longrightarrow & \mathcal{D}_{\mathcal{Z}_1 \leftarrow \mathcal{X}^b}^\dagger \longrightarrow 0 \end{array}$$

where the exact sequences are induced by (\star) above. The factorization $\mathcal{D}_{\mathcal{X}^b}^\dagger \rightarrow \mathcal{D}_{\mathcal{X}}^\dagger \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(\mathcal{Z}_1)$ is defined so that the diagram is commutative. We can check that the middle vertical arrow is given by $\frac{dt_1}{t_1} \otimes Q^b \mapsto dt_1 \otimes \frac{1}{t_1} Q^b t_1 \otimes \frac{1}{t_1}$, where we denote by $\frac{1}{t_1}$ the homomorphism which sends the dual of dt_1 to $\frac{dt_1}{t_1}$, and the left vertical arrow is given by $P^b \mapsto P^b t_1 \otimes \frac{1}{t_1}$, hence this is the canonical inclusion $\mathcal{D}_{\mathcal{X}^b}^\dagger \rightarrow \mathcal{D}_{\mathcal{X}}^\dagger \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(\mathcal{Z}_1)$. By applying the functor $-\otimes_{\mathcal{D}_{\mathcal{X}^b}^\dagger}^{\mathbb{L}} \mathcal{F}$ to this diagram we get

$$\begin{array}{ccccc} [u_+(\mathcal{F}) \xrightarrow{\nabla} \Omega_{\mathcal{X}/\mathcal{Z}_1}^1 \otimes_{\mathcal{O}_{\mathcal{X}}} u_+(\mathcal{F})] & \xrightarrow{\sim} & \mathcal{D}_{\mathcal{Z}_1 \leftarrow \mathcal{X}}^\dagger \otimes_{\mathcal{D}_{\mathcal{X}}^\dagger}^{\mathbb{L}} \mathcal{D}_{\mathcal{X} \leftarrow \mathcal{X}^b}^\dagger \otimes_{\mathcal{D}_{\mathcal{X}^b}^\dagger}^{\mathbb{L}} \mathcal{F} \\ \uparrow & & \sim \uparrow \\ [\mathcal{F} \xrightarrow{\nabla^b} \Omega_{\mathcal{X}^b/\mathcal{Z}_1}^1 \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}] & \xrightarrow{\sim} & \mathcal{D}_{\mathcal{Z}_1 \leftarrow \mathcal{X}^b}^\dagger \otimes_{\mathcal{D}_{\mathcal{X}^b}^\dagger}^{\mathbb{L}} \mathcal{F}. \end{array}$$

Then, we get that the isomorphism of Lemma 3.5.6 is given by the diagram:

$$\begin{array}{ccccccc} \mathcal{E} & \xrightarrow{\partial_1} & \mathcal{E} & \longrightarrow & \text{coker } \partial_1(\mathcal{E}) & \longrightarrow & 0 \\ \uparrow & & \uparrow \frac{1}{t_1} & & \sim \uparrow \frac{1}{t_1} & & \\ \mathcal{F} & \xrightarrow{t_1 \partial_1} & \mathcal{F} & \longrightarrow & \text{coker } t_1 \partial_1(\mathcal{F}) & \longrightarrow & 0, \end{array}$$

By applying the functor $\text{coker } t_1 \partial_1$ to (3.5.2.1), we get the commutativity of the right square of the following canonical diagram

$$\begin{array}{ccccccc} F_{\mathcal{Z}_1}^* \text{coker } \partial_1(\mathcal{E}) & \xrightarrow{\sim} & \text{coker } \partial_1'(\mathcal{E}') & \xrightarrow{\sim} & \text{coker } \partial_1(F_{\mathcal{X}}^* \mathcal{E}) & \xrightarrow[\phi]{\sim} & \text{coker } \partial_1(\mathcal{E}) \\ \uparrow \frac{1}{t_1} & & \uparrow \frac{1}{t_1} & & \uparrow \frac{1}{t_1} & & \uparrow \frac{1}{t_1} \\ F_{\mathcal{Z}_1}^* \text{coker } t_1 \partial_1(\mathcal{F}) & \xrightarrow{\sim} & \text{coker } t_1 \partial_1'(\mathcal{F}') & \xrightarrow{\sim} & \text{coker } t_1 \partial_1(F_{\mathcal{X}}^* \mathcal{F}) & \xrightarrow[\phi]{\sim} & \text{coker } t_1 \partial_1(\mathcal{F}) \end{array}$$

This diagram is commutative. Indeed, for the left square, use (3.5.3.3) and (3.5.4.3), and for the other squares, we leave the verification to readers. ■

Lemma 3.5.7. *The canonical morphism $\operatorname{coker} {}_{t_1\partial_1}(\mathcal{F})(1) \rightarrow \operatorname{coker} N_{1,\mathcal{F}}$ is compatible with Frobenius.*

Proof. Consider the diagram:

$$\begin{array}{ccccccc} F_{\mathcal{Z}_1}^* \operatorname{coker} {}_{t_1\partial_1}(\mathcal{F}) & \xrightarrow{\sim} & \operatorname{coker} {}_{t'_1\partial'_1}(\mathcal{F}') & \xrightarrow[\sim]{q \cdot \operatorname{adj}} & \operatorname{coker} {}_{t_1\partial_1}(F_{\mathcal{X}}^* \mathcal{F}) & \xrightarrow[\sim]{\phi_{\mathcal{F}}} & \operatorname{coker} {}_{t_1\partial_1}(\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F_{\mathcal{Z}_1}^* \operatorname{coker} N_{1,\mathcal{F}} & \xrightarrow{\sim} & \operatorname{coker} F_{\mathcal{Z}_1}^*(N_{1,\mathcal{F}}) & \xrightarrow[\sim]{q} & \operatorname{coker} (N_{1,F_{\mathcal{X}}^* \mathcal{F}}) & \xrightarrow[\sim]{\phi_{\mathcal{H}}} & \operatorname{coker} N_{1,\mathcal{F}}, \end{array}$$

where the homomorphism q is the one induced by q times the canonical isomorphism $F_{\mathcal{Z}_1}^* \mathcal{H} \xrightarrow{\sim} i_1^*(F_{\mathcal{X}}^* \mathcal{F})$, and the second vertical arrow is induced by the canonical isomorphism $i_1^* \mathcal{F}' \xrightarrow{\sim} F_{\mathcal{Z}_1}^* \mathcal{H}$. We can check easily that the middle square is commutative (cf. (3.5.4.3) and (3.2.8.2)). The right square is commutative by functoriality and the left square by flatness of $F_{\mathcal{Z}_1}$. ■

Lemma 3.5.8. *Consider the commutative square of coherent $\mathcal{D}_{\mathcal{Z}_1, \mathbb{Q}}^\dagger$ -modules*

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{a_1} & \mathcal{E}'_1 \\ \downarrow f & & \downarrow f' \\ \mathcal{E}_2 & \xrightarrow{a_2} & \mathcal{E}'_2, \end{array}$$

whose homomorphism on the top is surjective. We suppose these modules have a Frobenius structure such that f, a_1, a_2 commute with Frobenius. Then so is f' .

Proof. The verification is left to the reader. ■

Proposition 3.5.9. *The isomorphisms*

$$\ker(N_{1,\mathcal{F}}) \xrightarrow{\sim} g_{1+} \ker(\theta_{\mathcal{E}}), \quad \operatorname{coker}(N_{1,\mathcal{F}}) \xrightarrow{\sim} g_{1+} \operatorname{coker}(\theta_{\mathcal{E}}),$$

which are constructed by composing (3.4.10.1) and (3.4.9.2), commute with Frobenius.

Proof. First, let us prove the the second isomorphism. Consider the following diagram:

$$\begin{array}{ccccccc} \mathcal{H}^0 g_{1+j!}(\mathcal{E}) & \xrightarrow{\theta_{\mathcal{E}}} & \mathcal{H}^0 g_{1+} \mathcal{E} & \longrightarrow & \mathcal{H}^0 g_{1+} \operatorname{coker} \theta_{\mathcal{E}} & \longrightarrow & 0 \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \\ \mathcal{H}^0 g_{1+} \circ u_+(\mathcal{F}(-\mathcal{Z}_1)) & \xrightarrow{\epsilon_{\mathcal{F}}} & \mathcal{H}^0 g_{1+} \circ u_+(\mathcal{F}) & \longrightarrow & \mathcal{H}^0 g_{1+}(\operatorname{coker}(\epsilon_{1,\mathcal{F}})) & \longrightarrow & 0, \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \\ \operatorname{coker} {}_{t_1\partial_1}(\mathcal{F}(-\mathcal{Z}_1)) & \xrightarrow{\alpha_{\mathcal{F}}} & \operatorname{coker} {}_{t_1\partial_1}(\mathcal{F})(1) & \longrightarrow & \operatorname{coker} N_{1,\mathcal{F}} & \longrightarrow & 0, \end{array}$$

where the bottom left square is constructed in the proof of (3.4.10.1). This diagram is commutative, indeed the left above square is commutative by applying the functor $\mathcal{H}^0 g_{1+}$ to (3.4.9.1). Since \mathcal{X} is affine, the functors $\operatorname{coker} {}_{t_1\partial_1}$ and $\mathcal{H}^0 g_{1+}$ are right exact, so the horizontal sequences are exact. Now, the horizontal morphisms on the top commute with Frobenius. The composition of the middle vertical homomorphisms $\mathcal{H}^0 g_{1+} \mathcal{E} \xrightarrow{\sim} \operatorname{coker} {}_{t_1\partial_1}(\mathcal{F})(1)$ is compatible by (3.5.3.2) and Lemma 3.5.6, and the right horizontal homomorphism on the bottom as well by Lemma 3.5.7. We conclude the proof by using Lemma 3.5.8.

Now, let us deduce the second isomorphism from the first one: by [Abe13, 3.12], (3.2.4.1), and Corollary 1.4.3 we get

$$\begin{aligned} \mathbb{D}_{\mathcal{X}} i_{1+} \ker N_{1,\mathcal{F}} & \xrightarrow{\sim} i_{1+} \mathbb{D}_{\mathcal{Z}_1} \ker N_{1,\mathcal{F}} \xrightarrow{\sim} i_{1+} (\ker N_{1,\mathcal{F}})^\vee (-d+1) \xrightarrow{\sim} i_{1+} \operatorname{coker} (N_{1,\mathcal{F}}^\vee) (-d+1) \\ & \xrightarrow{\sim} i_{1+} \operatorname{coker} (N_{1,\mathcal{F}^\vee}) (-d) \xrightarrow{\sim} \operatorname{coker} (\theta_{\mathcal{E}^\vee}) (-d) \xrightarrow{\sim} \operatorname{coker} (\theta_{\mathbb{D}_{\mathcal{X}, \mathcal{Z}_1} \mathcal{E}}) \xrightarrow{\sim} \mathbb{D}_{\mathcal{X}} \ker(\theta_{\mathcal{E}}), \end{aligned}$$

and the proposition follows. ■

Corollary 3.5.10. *We consider the situation in paragraph 3.4.12, and D_1 is not empty anymore. The isomorphisms (3.4.12.1) are compatible with Frobenius.*

Proof. From Lemma 3.4.13, it is enough to check it outside D_1 , and we can suppose D_1 to be empty. By the same reason given at the first step of the proof of Lemma 3.4.7, we can suppose $\underline{f} = \text{id}$. Since the compatibility is local on \mathcal{X} or Z_1 , we can suppose that there exists a étale map $h: \mathcal{X} \rightarrow \hat{\mathbb{A}}_{Z_1}^1$ such that g_1 is the composition of h with the projection $\hat{\mathbb{A}}_{Z_1}^1 \rightarrow Z_1$ and that the composition of i_1 with h is the zero section. Since by construction, outside of D_1 , the isomorphisms of (3.4.12.1) are equal to that of Proposition 3.5.9, the claim follows by this proposition. \blacksquare

3.6 Stability of mixedness for a unipotent F -isocrystal

3.6.1. Let X be a smooth scheme, Z be a strict normal crossing divisor of X , and put $Y := X \setminus Z$. Let $\{Z_i\}_{i \in I}$ be the set of irreducible components of Z . For $J \subset I$, we put $Z_J := \bigcap_{i \in J} Z_i$. Then we define

$$Z_{(0)} := X, \quad Z_{(k)} := \bigcup_{\#J=k} Z_J, \quad Z_{(k)}^\circ := Z_{(k)} \setminus Z_{(k+1)}.$$

Then $\{Z_{(k)}^\circ\}_{0 \leq k \leq r}$ is a smooth stratification of X . This stratification is denoted by $\text{Strat}_Z(X)$.

3.6.2. Let (Y, X, \mathcal{P}) be a frame such that X is smooth and there exists a strict normal crossing divisor Z of X such that $Y := X \setminus Z$. We define the category $\text{ULNM}(Y, \mathcal{P}/K)$ to be the full subcategory of $F\text{-Isoc}^{\dagger\dagger}(Y, \mathcal{P}/K)$ consisting of \mathcal{E} such that: 1. \mathcal{E} is an ι -mixed unipotent F -isocrystal, and 2. \mathcal{E} is the restriction of an overconvergent F -isocrystal on Y . Our main theorem of this section is as follows:

Theorem. *Let $j = (\star, \text{id}, \text{id}): (Y, X, \mathcal{P}) \rightarrow (X, X, \mathcal{P})$ be the morphism of frames.*

(i) Suppose X proper. For any $\mathcal{E} \in \text{ULNM}(Y, \mathcal{P}/K)$, then $j_!(\mathcal{E})|_W$ is an ι -mixed F -isocrystal for any stratum W of the stratification $\text{Strat}_Z(X)$. In particular, $j_!(\mathcal{E})$ is an ι -mixed F -complex.

(ii) Suppose Z smooth. If \mathcal{E} is ι -pure of weight w and if $j_{1+}(\mathcal{E})$ is ι -mixed, then $j_{1+}(\mathcal{E})$ is ι -pure of weight w .

Proof. Since \mathcal{E} is an ι -mixed unipotent F -isocrystal, there exists a unique convergent log-isocrystal \mathcal{F} on (X, Z) which induces \mathcal{E} . By dévissage, we can assume that \mathcal{E} is ι -pure of weight w .

Let us start to show (i). We proceed by induction on r . For $r = 0$, we put $Z = \emptyset$. This case is obvious, so we can suppose that the proposition is checked for $r' < r$. Let $D_1 := Z_2 \cup \dots \cup Z_r$, $Y_1 := Z_1 \setminus D_1$, and $j_1: Y_1 \rightarrow Z_1$ be the open immersion. We note that in the situation of §3.1, the nilpotent homomorphism $N_{1, \mathcal{F}}$ (and then $({}^\dagger D_1)(N_{1, \mathcal{F}})$) does not depend on the choice of local coordinates. Thus, by gluing, we may construct the nilpotent homomorphism denoted, by abuse of notation, by $({}^\dagger D_1)(N_{1, \mathcal{F}})$ of $F\text{-Isoc}^{\dagger\dagger}(Y_1, \mathcal{P}/K)$. Hence, we get the isocrystals $\mathcal{E}_1 := \ker({}^\dagger D_1)(N_{1, \mathcal{F}})$ and $\mathcal{E}_2 := \text{coker}({}^\dagger D_1)(N_{1, \mathcal{F}})$ in $F\text{-Isoc}^{\dagger\dagger}(Y_1, \mathcal{P}/K)$, which are endowed with monodromy filtration M . Then $\text{gr}_i^M(\mathcal{E}_1)$, $\text{gr}_i^M(\mathcal{E}_2)$ are ι -pure. Indeed, by Remark 2.1.10, the verification is local, and we may reduce to the situation of Theorem 3.3.3. Then by this theorem the purity follows. Hence, by induction hypothesis, $j_{1!}(\mathcal{E}_1)|_W$ and $j_{1!}(\mathcal{E}_2)|_W$ are in $D_{\text{isoc}, m}^b(W, \mathcal{P}/K)$ for any stratum W of $\text{Strat}_{D_1}(Z_1)$.

On the other hand, for any stratum W of $\text{Strat}_{D_1}(Z_1)$, we have isomorphisms for any $k \in \mathbb{Z}$

$$\mathcal{H}_t^k(\mathcal{H}_{Z_1}^{\dagger 0}(j_!(\mathcal{E}))|_W) \cong \mathcal{H}_t^k(j_1(\mathcal{E}_1)|_W), \quad \mathcal{H}_t^k(\mathcal{H}_{Z_1}^{\dagger 1}(j_!(\mathcal{E}))|_W) \cong \mathcal{H}_t^k(j_1(\mathcal{E}_2)|_W)$$

Indeed, since both right sides of both isomorphisms are isocrystals on W , by using Kedlaya's fully faithfulness theorem, we may assume that we are in the situation of §3.1. Thus, by Corollary 3.5.10, the isomorphisms follows. Combining with what we have proven, we get that $\mathcal{H}_{Z_1}^{\dagger k}(j_!(\mathcal{E}))|_W$ ($k = 0, 1$) is in $D_{\text{isoc}, m}^b(W, \mathcal{P}/K)$. Since we have the triangle

$$\mathcal{H}_{Z_1}^{\dagger 0}(j_!(\mathcal{E}))|_W \rightarrow \mathbb{R}\Gamma_{Z_1}^{\dagger}(j_!(\mathcal{E}))|_W \rightarrow \mathcal{H}_{Z_1}^{\dagger 1}(j_!(\mathcal{E}))|_W \xrightarrow{+},$$

we conclude by Lemma 2.1.12 that $j_!(\mathcal{E})\|_W \cong \mathbb{R}\Gamma_{Z_1}^\dagger(j_!(\mathcal{E}))\|_W$ is in $D_{\text{isoc},m}^b(W, \mathcal{P}/K)$ for any stratum W in $\text{Strat}_{D_1}(Z_1)$.

By replacing Z_1 by Z_2, \dots, Z_r and D_1 by $D_i := \bigcup_{j \neq i} Z_j$, we get the same result. By construction, any stratum of $\text{Strat}_Z(X)$ except for $Z_{(0)}^\circ = Y$ is a disjoint union of strata in $\text{Strat}_{D_i}(Z_i)$. Thus the theorem follows.

Let us show (ii). Since the verification is local, we may assume that we are in the situation of §3.1. Consider the exact sequence:

$$0 \rightarrow j_{!+}(\mathcal{E}) \rightarrow j_+(\mathcal{E}) \rightarrow \mathcal{H}_Z^{\dagger 1}(j_!(\mathcal{E})) \rightarrow 0.$$

By Remark 3.3.4, we get that $\mathcal{H}_Z^{\dagger 1}(j_!(\mathcal{E}))$ is ι -mixed of weight $\geq w+1$. Since $j_+(\mathcal{E})$ is ι -mixed of weight $\geq w$ by Lemma 2.2.6, this implies that $j_{!+}(\mathcal{E})$ is ι -mixed of weight $\geq w$. By Corollary 1.4.3 and Lemma 2.1.6.3, we get that $j_{!+}(\mathcal{E})$ is ι -mixed of weight $\leq w$. ■

Remark 3.6.3. The second condition in the definition of $\text{ULNM}(Y, \mathcal{P}/K)$ is used to apply Theorem 3.3.3. If we remove this condition, we do not know if Theorem 3.6.2 holds or not.

4 Theory of weights

Throughout this section, we consider situation (B) in Notation and convention.

4.1 Stability of mixedness

4.1.1. Let $\star \in \{\emptyset, \leq w, \geq w\}$. Let Y be a realizable variety and $\mathcal{E} \in F\text{-}D_{\text{ovhol}}^b(Y/K)$. We say that \mathcal{E} *satisfies the condition* $(\text{SQ})_\star$ if the following holds:

$(\text{SQ})_\star$ For any integer i , any subquotient of $\mathcal{H}_t^i(\mathcal{E})$ is ι -mixed of weight $\star + i$.

Theorem 4.1.2. *Let $f: X \rightarrow Y$ be a morphism of realizable varieties.*

- (i) *For any $\mathcal{E} \in F\text{-}D_{\geq w}^b(X/K)$, we have $f_+(\mathcal{E}) \in F\text{-}D_{\geq w}^b(Y/K)$.*
- (ii) *The dual functor \mathbb{D}_X exchanges $F\text{-}D_{\geq w}^b(X/K)$ and $F\text{-}D_{\leq -w}^b(X/K)$.*
- (iii) *The condition (SQ) holds for any F -complex in $F\text{-}D_m^b(X/K)$.*

Remark. See Theorem 4.2.3 for more complete results for (iii).

Proof. **0) Preliminaries.**

a) *Déviage in \mathcal{E} :* Let $\mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \xrightarrow{+}$ be an exact triangle of $F\text{-}D_{\geq w}^b(X/K)$. By Lemma 2.2.7, if the part (i) and (ii) of the theorem holds for \mathcal{E}' and \mathcal{E}'' , then so does for \mathcal{E} .

b) *Déviage in X :* Let $j: U \hookrightarrow X$ be an open immersion and $i: X \setminus U \hookrightarrow X$ be the complement. Then if the theorem holds for $f \circ j$ and $f \circ i$ then so does for f . This follows from Lemma 2.2.7.

c) *Déviage in Y :* Let V be a open subvariety of Y and $W := Y \setminus V$. If the theorem holds for $f': f^{-1}(V) \rightarrow V$ and $f'': f^{-1}(W) \rightarrow W$ then so does for f . Indeed, by Lemma 2.2.6, the theorem holds for the morphisms $f^{-1}(V) \xrightarrow{f'} V \hookrightarrow Y$ and $f^{-1}(W) \xrightarrow{f''} W \hookrightarrow Y$, and by ii) the claim follows.

d) Let $\mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \xrightarrow{+}$ be a triangle, and assume that (SQ) holds for \mathcal{E}' and \mathcal{E}'' . Then (SQ) holds for \mathcal{E} as well.

1) By Lemma 2.2.6, the theorem holds when X is of dimension 0 and we may assume that $\dim(Y) \leq \dim(X)$. We proceed by induction on the dimension of X . Assume the theorem holds for $\dim(X) < n$. We will show the theorem for $\dim(X) = n$. Let $\mathcal{E} \in F\text{-}D_{\geq w}^b(X/K)$.

2) a) Let us show (i) in the case where f is quasi-finite. If f is an immersion, then by Lemma 2.2.6 the theorem holds. Hence, by using EGA IV, Theorem 8.12.6, we may assume that f is finite. By using 0.a) and 0.b), we may assume that X is smooth, $\mathcal{E} \in F\text{-}\text{Isoc}^{\dagger\dagger}(X/K)$ and it is ι -pure of weight w . Moreover, thanks to 0.c) and the induction hypothesis, we may further suppose that Y is smooth and that $f_+(\mathcal{E}) \in F\text{-}D_{\text{isoc}}^b(Y/K)$. Then

it is sufficient to check that for every closed point y of Y , $i_y^! \circ f_+(\mathcal{E})$ is ι -pure of weight w where $i_y: \{y\} \hookrightarrow Y$ is the closed immersion as usual. Hence, by using a base change theorem 1.3.14, we reduce to the case where $\dim X = \dim Y = 0$, and the theorem follows.

b) Assume f is a universal homeomorphism. From 2.a) and Proposition 2.2.4, f_+ and $f^!$ preserve the mixedness $\geq w$. Thus, by Proposition 1.3.12, the functors f_+ and $f^!$ induce canonical equivalences of categories between $F\text{-}D_{\geq w}^b(X/K)$ and $F\text{-}D_{\geq w}^b(Y/K)$.

3) Let us show (ii) and (iii).

a) Let us show (ii) for $\dim(X) = n$. We may assume X to be proper. By dévissage, it suffices to show that for an open affine smooth subscheme $j: U \hookrightarrow X$ and an ι -mixed overconvergent F -isocrystal \mathcal{E} on U , $j_!(\mathcal{E})$ is ι -mixed. By using Kedlaya's semistable reduction theorem [Ked11], there exists a generically finite étale morphism $g: X' \rightarrow X$ such that X' is projective smooth, $Z' := X' \setminus g^{-1}(U)$ is a strictly normal crossing divisor, and $g_U^*(\mathcal{E})$ is a unipotent F -isocrystal where $g_U: g^{-1}(U) \rightarrow U$. There exists an open subscheme $V \subset U$ such that $g_V: V' := g^{-1}(V) \rightarrow V$ is finite étale. For $\star \in \{U, V\}$, let $j_\star: \star \hookrightarrow X$, $j'_\star: g^{-1}(\star) \hookrightarrow X'$. Now, for a ι -mixed F -module \mathcal{F} on X' , we claim that $g_+(\mathcal{F})$ is ι -mixed. Indeed, consider the triangle $\mathbb{R}\Gamma_{X' \setminus V'}^\dagger(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow j'_{V'+} \mathcal{F}|_{V'} \xrightarrow{+}$. By induction hypothesis, $g_+(\mathbb{R}\Gamma_{X' \setminus V'}^\dagger(\mathcal{F}))$ is ι -mixed, and since g_V is finite étale, by 2.a), $g_+ j'_{V'+} \mathcal{F}|_{V'}$ is ι -mixed as well, and the claim follows.

By Theorem 3.6.2, $j'_{U!}(g_U^* \mathcal{E})$ is ι -mixed. By using the induction hypothesis, this is equivalent to the property that $j'_{V!}(g_U^*(\mathcal{E})|_{V'})$ is ι -mixed as well. Now, by the claim, so is $g_+ j'_{V!}(g_U^*(\mathcal{E})|_{V'})$. Since it contains $j_{V!}(\mathcal{E}|_V)$ as a direct factor, then $j_{V!}(\mathcal{E}|_V)$ is ι -mixed, and using the induction hypothesis again, we get that $j_!(\mathcal{E})$ is ι -mixed.

b) Let us show (iii) for $\dim(X) = n$. For \mathcal{E} , there exists an open subscheme $j: U \hookrightarrow X$ such that U is smooth and the complement of a divisor in X and $\mathcal{E}|_U$ is an isocrystal. By considering the localization triangle, 0.d), and induction hypothesis, it suffices to show (SQ) for $j_+(\mathcal{F})$ where \mathcal{F} is an irreducible overconvergent F -isocrystal on U . Then, by (1.4.4.1), we get the exact sequence

$$0 \rightarrow j_{!+}(\mathcal{F}) \rightarrow j_+(\mathcal{F}) \rightarrow \mathbb{R}^1 \Gamma_Z^\dagger(j_!(\mathcal{F})) \rightarrow 0.$$

We have already shown at 2.a) that $j_+(\mathcal{F})$ and $j_!(\mathcal{F})$ are ι -mixed. Thus, $\mathbb{R}^1 \Gamma_Z^\dagger(j_!(\mathcal{F}))$ is ι -mixed, and $\mathbb{R}^1 \Gamma_Z^\dagger(j_!(\mathcal{F}))$ is ι -mixed by the induction hypothesis on (SQ). Thus, by 0.d), $j_{!+}(\mathcal{F})$ is ι -mixed. Since $j_{!+}(\mathcal{F})$ is irreducible by Proposition 1.4.7, this satisfies (SQ). Thus, (SQ) for $j_+(\mathcal{F})$ follows by 1.d).

4) It remains to check (i). By using 0), 2.a), and Noether's normalization theorem, we may assume that f is a smooth morphism of relative dimension 1, X is smooth, and $\mathcal{E} \in F\text{-Isoc}^{\dagger\dagger}(X/K)$. By 0.c), we may assume that Y is integral, affine and smooth. By using Kedlaya's semistable reduction theorem, we can suppose that there exist a projective surjective morphism $g: X' \rightarrow X$, an open immersion $X' \hookrightarrow \overline{X}'$, an open dense subvariety U of X such that the induced morphism $g': U' := g^{-1}(U) \rightarrow U$ is finite étale, \overline{X}' is projective smooth, $Z' := \overline{X}' \setminus X'$ is a strict normal crossing divisor \overline{X}' and $g^*(\mathcal{E})$ is a unipotent overconvergent F -isocrystal. Let \mathcal{K} be the function field of Y . Since $U'_\mathcal{K}$ is a smooth curve over $\text{Spec}(\mathcal{K})$, we have the canonical smooth compactification $C'_\mathcal{K}$ over \mathcal{K} . Since $\overline{X}'_\mathcal{K}$ is projective, there exists a unique morphism $\alpha_\mathcal{K}: C'_\mathcal{K} \rightarrow \overline{X}'_\mathcal{K}$, which is nothing but the normalization morphism, that extends the inclusion $U'_\mathcal{K} \rightarrow \overline{X}'_\mathcal{K}$. By using EGA IV, Theorem 8.8.2 and shrinking Y , we may assume that $C'_\mathcal{K}$ comes from a variety C' which is projective smooth of relative dimension 1 over Y' . By shrinking Y further, we can assume that $\alpha_\mathcal{K}$ (resp. the open immersion $U'_\mathcal{K} \hookrightarrow C'_\mathcal{K}$) comes from a projective morphism $\alpha: C' \rightarrow \overline{X}'$ (resp. an open immersion $j': U' \hookrightarrow C'$ such that $\alpha \circ j'$ is the canonical open immersion), and that the complement $D' := C' \setminus U'$ is smooth. Let $\mathcal{E}_U \in F\text{-Isoc}^{\dagger\dagger}(U/K)$ be the overconvergent F -isocrystal on U induced by \mathcal{E} . Since $g^*(\mathcal{E})$ has unipotent monodromy along Z' , $g^*(\mathcal{E})|_{U'}$, the object of $F\text{-Isoc}^{\dagger\dagger}(U'/K)$ induced by restriction of $g^*(\mathcal{E})$, comes from a convergent isocrystal \mathcal{G} over the log-scheme (\overline{X}', Z') . Hence $g'^*(\mathcal{E}_U)$ comes from a convergent log-isocrystal on (\overline{X}', Z') . Now, let $\alpha^\#: (C', D') \rightarrow (\overline{X}', Z')$ be the morphism of log-schemes induced by α . Then we can say that $g'^*(\mathcal{E}_U)|_{(U', C')}$ (caution that this is the isocrystal in $F\text{-Isoc}^{\dagger\dagger}(U', C'/K)$ defined by the

restriction and (U', C') is not a log-scheme!) comes from the convergent log-isocrystal $\alpha^{\#*}(\mathcal{G})$ on (C', D') . Let $f_U: U \rightarrow Y$ be the morphism induced by f . Since \mathcal{E}_U is a direct factor of $g'_+ g'^*(\mathcal{E}_U)$, $f_{U+}(\mathcal{E}_U)$ is a direct factor of $(f_U \circ g')_+ g'^*(\mathcal{E}_U)$. Hence, by using 0.b), we can suppose that there exists the following commutative diagram of varieties

$$\begin{array}{ccc} D & \xrightarrow{i} & \overline{X} \xleftarrow{j} X \\ & \searrow h & \swarrow f \\ & & Y \end{array}$$

where h is projective smooth purely of relative dimension 1, j is an open immersion, i is the induced closed immersion, where D is a smooth divisor of \overline{X} and \mathcal{E} is ι -pure of weight w having unipotent monodromy along D . Now, there exists a finite radical extension \mathcal{L} of K such that the morphism $D_{\mathcal{L}} \rightarrow \text{Spec}(\mathcal{L})$ is finite étale. Let $a: \tilde{Y} \rightarrow Y$ be the normalization of Y in \mathcal{L} . By 2.b), we may take the pull-back by a , and shrinking Y further, we may assume moreover that $h \circ i$ is finite étale. By shrinking Y once again, we may assume $h_+(j_+(\mathcal{E})) \in F\text{-}D_{\text{isoc}}^b(Y/K)$.

5) Let us finish the proof of (i). By 3.a), we know that $j_!(\mathcal{E})$ is ι -mixed. Thus, $\mathbb{R}\Gamma_D^{\dagger}(j_!(\mathcal{E}))$ is ι -mixed as well, and since (SQ) holds for this F -complex, we get that $\mathbb{R}^1\Gamma_D^{\dagger}(j_!(\mathcal{E}))$ is ι -mixed. Thus, we get that $j_{!+}(\mathcal{E})$ is ι -mixed. Moreover, by Theorem 3.6.2 (ii), $j_{!+}(\mathcal{E})$ is ι -pure of weight w . Now, let us show that $h_+(j_{!+}(\mathcal{E}))$ is ι -pure of weight w . For this, it suffices to show that for an ι -pure F -module \mathcal{F} of weight w on \overline{X} such that $h_+(\mathcal{F}) \in F\text{-}D_{\text{isoc}}^b(Y/K)$, $h_+(\mathcal{F})$ is ι -pure of weight w . Let y be a closed point of Y , and $i_y: \{y\} \hookrightarrow Y$ be the closed immersion. We put $i'_y: h^{-1}(y) \hookrightarrow \overline{X}$ the closed immersion, and $h'_y: h^{-1}(y) \rightarrow \{y\}$ the proper smooth curve. By base change theorem 1.3.14, we have $i_y^! h_+(\mathcal{F}) \cong h'_{y+} i_y'^!(\mathcal{F})$. Since $i_y'^!(\mathcal{F})$ is ι -mixed of weight $\geq w$, by Proposition 2.2.4, $i_y^! h_+(\mathcal{F})$ is ι -mixed of weight $\geq w$. Since h is proper, $h_! \cong h_+$ by paragraph 1.3.14. By 3.a), since $\mathbb{D}_{\overline{X}}(\mathcal{F})$ is ι -pure of weight $-w$, we get that $i_y^+ h_+(\mathcal{F})$ is ι -mixed of weight $\leq w$, and we conclude that $h_+(\mathcal{F})$ is ι -pure of weight w .

Finally, consider the exact sequence

$$0 \rightarrow j_{!+}\mathcal{E} \rightarrow j_+\mathcal{E} \rightarrow \mathcal{H}_t^1 \mathbb{R}\Gamma_D^{\dagger}(j_!(\mathcal{E})) \rightarrow 0.$$

Since $j_{!+}(\mathcal{E})$ and $j_+(\mathcal{E})$ are ι -mixed of weight $\geq w$, so is $\mathcal{H}_t^1 \mathbb{R}\Gamma_D^{\dagger}(j_!(\mathcal{E}))$ by Lemma 2.2.7. By the finite étaleness of $h \circ i$, $h_+(\mathcal{H}_t^1 \mathbb{R}\Gamma_D^{\dagger}(j_!(\mathcal{E})))$ is ι -mixed of weight $\geq w$. Since we showed that $h_+(j_{!+}(\mathcal{E}))$ is ι -pure of weight w , we get that $f_+(\mathcal{E}) \cong h_+(j_+(\mathcal{E}))$ is ι -mixed of weight $\geq w$, which concludes the proof. \blacksquare

4.1.3. Let us sum up the results.

Theorem. *Let $f: X \rightarrow Y$ be a morphism of realizable varieties. We get:*

1. $f_!$ and f^+ preserve $D_{\leq w}^b$;
2. $f^!$ and f_+ preserve $D_{\geq w}^b$;
3. $\tilde{\otimes}_Y$ sends $D_{\geq w}^b \times D_{\geq w'}^b$ to $D_{\geq w+w'}^b$, and \otimes_Y sends $D_{\leq w}^b \times D_{\leq w'}^b$ to $D_{\leq w+w'}^b$;
4. \mathbb{D}_Y exchanges $D_{\leq w}^b$ and $D_{\geq -w}^b$.

4.2 Purity results

In this subsection, we prove the purity stability for intermediate extension. With this purity, we complete the theory of weights in p -adic cohomology theory. For the proof of Theorem 4.2.2, we follow the idea of Kiehl-Weissauer [KW01].

Lemma 4.2.1. *Let X be a realizable variety, and $\{X_t\}_{t \in I}$ be a set of divisors of X such that $X_t \cap X_{t'} = \emptyset$ for any $t \neq t'$ in I . Let $i_t: X_t \hookrightarrow X$ be the closed immersion. Then for any overholonomic F -module \mathcal{E} over X , the F -complex $i_t^! \mathcal{E}[1]$ is in fact a F -module (or equivalently $\mathcal{H}_t^0 i_t^! \mathcal{E} = 0$) for any but finitely many $t \in I$.*

Proof. We have an injection $i_{t+}\mathcal{H}_t^0(i_t^!\mathcal{E}) \hookrightarrow \mathcal{E}$. Since $i_{t+}\mathcal{H}_t^0(i_t^!\mathcal{E})$ is supported on X_t , different $t \in I$ give different F -submodules of \mathcal{E} . Since there are finitely many constituents of an overholonomic F -module, we conclude that $\mathcal{H}_t^0(i_t^!\mathcal{E}) = 0$ except for finitely many t in I . In this case, since X_t is assumed to be a divisor, $i_t^!\mathcal{E}$ is concentrated at degree 1, and the lemma follows. \blacksquare

Theorem 4.2.2. *Let X be a realizable variety, and $i: Z \hookrightarrow X$ be a closed subvariety whose complement is denoted by U . Let \mathcal{E} be an overholonomic F -module on X which is ι -mixed and such that $\mathcal{H}_t^0 i^!\mathcal{E} = 0$. If $\mathcal{E}|_U$ is ι -mixed of weight $\geq w$, so is \mathcal{E} .*

Proof. Since the verification is local, we may assume that X is affine. Take a closed embedding $X \hookrightarrow \mathbb{A}_k^n$ for some n . The assumption being stable under closed push-forward, we assume X to be \mathbb{A}_k^n from now on. Now, we will show the theorem using the induction on n . Assume that the theorem holds for $n < N$.

Let us, first, reduce to the case $\dim(Z) = 0$. Choose a linear projection $X = \mathbb{A}_k^N \rightarrow T := \mathbb{A}_k^1$. For $t \in |T|$, we denote $i_t: X_t := X \times_T \{t\} \hookrightarrow X$, the induced closed immersion. If there is no risk of confusion, we denote by $i: Z_t := Z \times_T \{t\} \hookrightarrow X_t$ and $i_t: Z_t := Z \times_T \{t\} \hookrightarrow Z$ the induced closed immersions. By Lemma 4.2.1, there exists a finite set $B \subset |T|$ of closed points such that, for any $t \notin B$, we have $\mathcal{H}_t^0 i_t^!\mathcal{E} = 0$. By using Lemma 4.2.1 once again, by enlarging B if necessary, we have $\mathcal{H}_t^0 i_t^!(\mathcal{H}_t^1 i^!\mathcal{E}) = 0$ for any $t \notin B$. Since the assumption shows that $\mathcal{H}_t^0 i^!\mathcal{E} = 0$, we get $\mathcal{H}_t^1(i_t^! i^!\mathcal{E}) = 0$ for any $t \notin B$. Since $i_t^! i^! \xrightarrow{\sim} i^! i_t^!$, this implies that $\mathcal{H}_t^0 i^! \mathcal{H}_t^1 i^!\mathcal{E} = 0$ for any $t \notin B$. Since $\mathcal{E}|_U$ is ι -mixed of weight $\geq w$, then $\mathcal{H}_t^1 i_t^!(\mathcal{E})|_{U_t}$ is ι -mixed of weight $\geq w + 1$, where $U_t := U \times_T \{t\}$. Hence, by induction, we get that $\mathcal{H}_t^1 i_t^!\mathcal{E}$ is ι -mixed of weight $\geq w + 1$, for any $t \notin B$. By applying the same argument for the other projection $\mathbb{A}_k^N \rightarrow \mathbb{A}_k^1$, we can check there exists a closed subscheme $Z' \subset Z$ of dimension 0 such that \mathcal{E} is ι -mixed of weight $\geq w$ on $X \setminus Z'$. By the transitivity and left exactness of $i^!$, we may replace Z by Z' .

Now, let us finish the proof by treating the case $\dim(Z) = 0$. Since the weight do not change by extension of K by Lemma 2.2.10, we may assume $\pi \in K$ to use geometric Fourier transform. We put $T_\pi(\mathcal{E}) := \mathcal{F}_\pi(\mathcal{E})[2]$ and $\mathcal{J}_\pi := \mathcal{K}_\pi[2]$, which makes the description slightly simpler because the weight of \mathcal{J}_π is 0. Let $x \in |\mathbb{A}_k^N|$, and $k(x)$ be its residue field, $\iota_x: \text{Spec } k(x) \hookrightarrow \mathbb{A}^N$, $\alpha_x: \mathbb{A}_x^N := \mathbb{A}^N \times_{\text{Spec } k} \text{Spec } k(x) \hookrightarrow \mathbb{A}^{2N}$, $i_x: Z_x := Z \times_{\text{Spec } k} \text{Spec } k(x) \hookrightarrow \mathbb{A}_x^N$ be the closed immersions, and $\mathcal{E}_x := \alpha_x^! \circ p_1^!(\mathcal{E})$ and $\rho_x: \text{Spec } k(x) \rightarrow \text{Spec } k$ the induced morphism. Since $p_1 \circ \alpha_x$ is finite and étale, $\mathcal{H}_t^0 i_x^!\mathcal{E}_x = 0$ and \mathcal{E}_x is ι -mixed of weight $\geq w$ outside Z_x . For a realizable variety $f: Y \rightarrow \text{Spec}(k)$ and an overholonomic F -complex \mathcal{F} over Y , we denote $\mathcal{H}_t^m(f_+(\mathcal{F}))$ by $H^m(Y, \mathcal{F})$. By (1.1.9.1) and the base change, we get:

$$\begin{aligned} H^m(\mathbb{A}_x^N, \mathcal{E}_x \widetilde{\otimes} \alpha_x^! \mathcal{J}_\pi) &\xrightarrow{\sim} H^m(\mathbb{A}_x^N, \alpha_x^!(p_1^!(\mathcal{E}) \widetilde{\otimes} \mathcal{J}_\pi)) \\ &\xrightarrow{\sim} \mathcal{H}_t^m(\rho_{x,+} \circ \iota_x^! \circ p_{2+}(p_1^!(\mathcal{E}) \widetilde{\otimes} \mathcal{J}_\pi)) \cong \mathcal{H}_t^m(\rho_{x,+} \circ \iota_x^!(T_\pi(\mathcal{E}))). \end{aligned} \quad (4.2.2.1)$$

Since $T_\pi(\mathcal{E})[-N]$ is an F -module by 2.3.4.1, we get $H^m(\mathbb{A}_x^N, \mathcal{E}_x \widetilde{\otimes} \alpha_x^! \mathcal{J}_\pi) = 0$ for any integer $m \notin [-N, 0]$ by (4.2.2.1). Moreover since $i_x^!(\mathcal{E}_x \widetilde{\otimes} \alpha_x^! \mathcal{J}_\pi) \xrightarrow{\sim} i_x^!(\mathcal{E}_x) \widetilde{\otimes} i_x^! \alpha_x^! \mathcal{J}_\pi$, $i_x^! \alpha_x^! \mathcal{J}_\pi$ is a F -module, and $\mathcal{H}_t^0 i_x^! \mathcal{E}_x = 0$, we get that $H^m(Z_x, i_x^!(\mathcal{E}_x \widetilde{\otimes} \alpha_x^! \mathcal{J}_\pi)) = 0$ for any integer $m \notin [1, N]$. Now, the exact sequence

$$\rightarrow H^m(Z_x, i_x^!(\mathcal{E}_x \widetilde{\otimes} \alpha_x^! \mathcal{J}_\pi)) \rightarrow H^m(\mathbb{A}_x^N, \mathcal{E}_x \widetilde{\otimes} \alpha_x^! \mathcal{J}_\pi) \rightarrow H^m(\mathbb{A}_x^N \setminus Z_x, \mathcal{E}_x \widetilde{\otimes} \alpha_x^! \mathcal{J}_\pi) \rightarrow$$

implies that

$$H^m(\mathbb{A}_x^N, \mathcal{E}_x \widetilde{\otimes} \alpha_x^! \mathcal{J}_\pi) \begin{cases} \cong H^m(\mathbb{A}_x^N \setminus Z_x, \mathcal{E}_x \widetilde{\otimes} \alpha_x^! \mathcal{J}_\pi) & \text{for } -N \leq m < 0 \\ \hookrightarrow H^0(\mathbb{A}_x^N \setminus Z_x, \mathcal{E}_x \widetilde{\otimes} \alpha_x^! \mathcal{J}_\pi) & \text{for } m = 0 \\ = 0 & \text{otherwise.} \end{cases}$$

By Theorem 4.1.3, $H^m(\mathbb{A}_x^N \setminus Z_x, \mathcal{E}_x \widetilde{\otimes} \alpha_x^! \mathcal{J}_\pi)$ is ι -mixed of weight $\geq w + m$ (recall that \mathcal{J}_π is ι -pure of weight 0), and thus from (4.2.2.1), $\iota_x^!(T_\pi(\mathcal{E}))$ is of weight $\geq w$. Hence, $T_\pi(\mathcal{E})$ is ι -mixed of weight $\geq w$. By Theorem 4.1.3 and 2.3.4.2, we note that $\mathcal{F} \in D_{\geq w}^b(\mathbb{A}^N)$ if and only if $T_\pi(\mathcal{F}) \in D_{\geq w}^b(\mathbb{A}^N)$, and the theorem follows. \blacksquare

Theorem 4.2.3. *Let X be a realizable variety, $\mathcal{E} \in F\text{-}D_{\text{overhol}}^b(X/K)$, and $\star \in \{\emptyset, \leq w, \geq w\}$. If \mathcal{E} is ι -mixed of weight \star , then \mathcal{E} satisfies the condition $(\text{SQ})_\star$.*

Proof. By Remark 2.2.3 (i) and duality, it is sufficient to consider the case where \star is $\geq w$. We prove the theorem by induction on the dimension of X . The case where $\dim(X) = 0$ is obvious. If we have a distinguished triangle $\mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \xrightarrow{+}$ in $F\text{-}D_{\text{ovhol}}^b(X/K)$ such that $(\text{SQ})_{\geq w}$ holds for \mathcal{E}' and \mathcal{E}'' , then \mathcal{E} satisfies $(\text{SQ})_{\geq w}$ as well. Hence, by dévissage, it suffices to show that for any smooth irreducible open subscheme $j: U \hookrightarrow X$ such that j is affine and for any irreducible overconvergent F -isocrystal \mathcal{E} on U which is ι -pure of weight w , the F -module $j_+\mathcal{E}$ satisfies $(\text{SQ})_{\geq w}$. For this, let \mathcal{F} be an overholonomic F -module on X defined by the exact sequence $0 \rightarrow j_{!+}\mathcal{E} \rightarrow j_+\mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$. Let $Z := X \setminus U$. By Theorem 4.1.3, the F -complex $\mathbb{R}\Gamma_Z^\dagger(j_!(\mathcal{E}))$ is ι -mixed. Since $\mathbb{R}\Gamma_Z^\dagger(j_+(\mathcal{E})) = 0$, one get $\mathcal{F} \xrightarrow{\sim} \mathcal{H}_t^1(\mathbb{R}\Gamma_Z^\dagger(j_!(\mathcal{E})))$, and the induction hypothesis shows that this satisfies the (SQ) property and *a fortiori* is ι -mixed. Hence $j_{!+}\mathcal{E}$ is ι -mixed. From Theorem 4.2.2, By Corollary 1.4.3 and Proposition 1.4.7, we get that $j_{!+}\mathcal{E}$ is irreducible and ι -pure of weight w , and hence satisfies $(\text{SQ})_{\geq w}$. Since $\mathcal{F} \xrightarrow{\sim} \mathcal{H}_t^1(\mathbb{R}\Gamma_Z^\dagger(j_+(\mathcal{E})))$, \mathcal{F} is ι -mixed of weight $\geq w + 1$. Thus, by induction hypothesis, \mathcal{F} satisfies $(\text{SQ})_{\geq w+1}$ and *a fortiori* $(\text{SQ})_{\geq w}$. Thus the theorem follows. \blacksquare

We see easily that the theorem leads to:

Corollary 4.2.4. *Let $j: U \hookrightarrow X$ be an open immersion of realizable varieties. The intermediate extension $j_{!+}$ sends an ι -pure F -module on U of weight w to an ι -pure F -module on X of weight w .*

4.3 Applications

This subsection is devoted to exhibit a few applications of the theory of weights. Let $*$ be either $\mathcal{D}_{X,\mathbb{Q}}^\dagger$ or $F\text{-}\mathcal{D}_{X,\mathbb{Q}}^\dagger$. We denote by Hom_* and Ext_*^1 the Hom group and Yoneda's Ext group of the abelian category of overholonomic $*$ -modules respectively.

Theorem 4.3.1 (Semi-simplicity of pure F -modules). *Let X be a realizable variety, and \mathcal{E} be an ι -pure F -module in $F\text{-}\text{Ovhol}(X/K)$. Then \mathcal{E} is semi-simple in $\text{Ovhol}(X/K)$ (not in $F\text{-}\text{Ovhol}(X/K)$).*

Proof. Let $\mathcal{F} \subset \mathcal{E}$ be the maximal semi-simple submodule in $\text{Ovhol}(X/K)$. Then we see that \mathcal{F} is stable under the Frobenius structure of \mathcal{E} , and the inclusion $\mathcal{F} \subset \mathcal{E}$ is in fact in $F\text{-}\text{Ovhol}(X/K)$. By Theorem 4.2.3, \mathcal{F} and \mathcal{E}/\mathcal{F} are ι -pure as well. This gives us $[\mathcal{E}] \in \text{Ext}_{F\text{-}\mathcal{D}_{X,\mathbb{Q}}^\dagger}^1(\mathcal{E}/\mathcal{F}, \mathcal{F})$. Now, by using Proposition A.5, Theorem 4.1.3, and Remark A.4, we have $\text{Ext}_{\mathcal{D}_{X,\mathbb{Q}}^\dagger}^1(\mathcal{E}/\mathcal{F}, \mathcal{F})^F = 0$. Since the canonical homomorphism $\alpha: \text{Ext}_{F\text{-}\mathcal{D}_{X,\mathbb{Q}}^\dagger}^1(\mathcal{E}/\mathcal{F}, \mathcal{F}) \rightarrow \text{Ext}_{\mathcal{D}_{X,\mathbb{Q}}^\dagger}^1(\mathcal{E}/\mathcal{F}, \mathcal{F})$ factors through $\text{Ext}_{\mathcal{D}_{X,\mathbb{Q}}^\dagger}^1(\mathcal{E}/\mathcal{F}, \mathcal{F})^F \rightarrow \text{Ext}_{\mathcal{D}_{X,\mathbb{Q}}^\dagger}^1(\mathcal{E}/\mathcal{F}, \mathcal{F})$, we obtain $\alpha([\mathcal{E}]) = 0$, which concludes the proof. \blacksquare

Lemma 4.3.2. *Let X be a realizable variety, and \mathcal{E}, \mathcal{F} be objects in $F\text{-}\text{Ovhol}(X/K)$. Then, we have the following short exact sequence:*

$$0 \rightarrow \text{Hom}_{\mathcal{D}_{X,\mathbb{Q}}^\dagger}(\mathcal{E}, \mathcal{F})^F \rightarrow \text{Ext}_{F\text{-}\mathcal{D}_{X,\mathbb{Q}}^\dagger}^1(\mathcal{E}, \mathcal{F}) \rightarrow \text{Ext}_{\mathcal{D}_{X,\mathbb{Q}}^\dagger}^1(\mathcal{E}, \mathcal{F})^F \rightarrow 0.$$

Proof. Let $0 \rightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{E} \rightarrow 0$ be an extension of $\mathcal{D}_{X,\mathbb{Q}}^\dagger$ -modules, and denote by $[\mathcal{G}] \in \text{Ext}_{\mathcal{D}_{X,\mathbb{Q}}^\dagger}^1(\mathcal{E}, \mathcal{F})$ the class defined by the extension. Let $\phi_{\mathcal{F}}: \mathcal{F} \rightarrow \mathcal{F}$ be the Frobenius structure of \mathcal{F} , and the same for $\phi_{\mathcal{E}}$. Then the class $F^*[\mathcal{G}]$ can be written as

$$0 \rightarrow \mathcal{F} \xrightarrow{\alpha \circ \phi_{\mathcal{F}}} \mathcal{G} \xrightarrow{\phi_{\mathcal{E}}^{-1} \circ \beta} \mathcal{E} \rightarrow 0.$$

The condition $F^*[\mathcal{G}] = [\mathcal{G}]$ means that there exists a dotted arrow making the diagram commutative:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{E} \longrightarrow 0 \\ & & \downarrow \phi_{\mathcal{F}} & \sim & \downarrow \text{dotted} & \sim & \downarrow \phi_{\mathcal{E}} \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{E} \longrightarrow 0. \end{array}$$

Thus the surjectivity of the homomorphism $\text{Ext}_{F\text{-}\mathcal{D}_{X,\mathbb{Q}}^\dagger}^1(\mathcal{E}, \mathcal{F}) \rightarrow \text{Ext}_{\mathcal{D}_{X,\mathbb{Q}}^\dagger}^1(\mathcal{E}, \mathcal{F})^F$ follows. Now, assume that the image of $[(\mathcal{G}, \phi_{\mathcal{G}})]$ is 0. This means that \mathcal{G} is split. Then the computation of the kernel is standard. (See [BBD82, proof of 5.1.2] for example.) ■

Lemma 4.3.3. *Let X be a realizable variety, and \mathcal{E}, \mathcal{F} be objects in $F\text{-Ovhol}(X/K)$. Assume that \mathcal{E}, \mathcal{F} are irreducible ι -pure F -modules such that $\text{wt}(\mathcal{F}) > \text{wt}(\mathcal{E})$. Then $\text{Ext}_{F\text{-}\mathcal{D}_{X,\mathbb{Q}}^\dagger}^1(\mathcal{E}, \mathcal{F}) = 0$.*

Proof. It suffices to show that the two outer modules appearing in the short exact sequence of Lemma 4.3.2 vanish. By Proposition A.5 and Theorem 4.1.3, we get the claim. ■

Theorem 4.3.4 (Weight filtration). *Let X be a realizable variety, and \mathcal{E} be an object of $F\text{-Ovhol}(X/K)$ which is ι -mixed. Then there exists a unique increasing filtration W on \mathcal{E} such that $\text{gr}_i^W(\mathcal{E})$ is purely of ι -weight i . Any homomorphism of ι -mixed F -modules on X is strictly compatible with the filtration.*

Proof. Using Lemma 4.3.3, we can follow the proof of [BBD82, 5.3.6]. ■

4.3.5. Let X be a realizable variety, and $\mathcal{E}, \mathcal{F} \in F\text{-}D_{\text{ovhol}}^b(X/K)$. By definition, we have the equality:

$$\text{Hom}_{F\text{-}D_{\text{ovhol}}^b(X/K)}(\mathcal{E}, \mathcal{F}) = \text{Hom}_{D_{\text{ovhol}}^b(X/K)}(\mathcal{E}, \mathcal{F})^F.$$

By Theorem 4.1.3 and Proposition A.5, we get that if $\mathcal{E} \in F\text{-}D_{\leq w}^b(X/K)$ and $\mathcal{F} \in F\text{-}D_{> w}^b(X/K)$ then

$$\text{Hom}_{F\text{-}D_{\text{ovhol}}^b(X/K)}(\mathcal{E}, \mathcal{F}) = 0. \quad (4.3.5.1)$$

Theorem 4.3.6 (Semi-simplicity). *Let X be a realizable variety, and \mathcal{E} be an ι -pure overholonomic F -complex on X . Then \mathcal{E} is isomorphic, in $D_{\text{ovhol}}^b(X/K)$, to $\bigoplus_{n \in \mathbb{Z}} \mathcal{H}_t^n(\mathcal{E})[n]$.*

Proof. Using (4.3.5.1), the proof is essentially the same as that of [BBD82, 5.4.5, 5.4.6]. ■

Remark 4.3.7. For a scheme X over a field, denote by $D_c^b(X)$ the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$ -complexes where ℓ is a prime number different from p . Let X be a variety over the finite field k . As pointed out in Remark 1.1.5, the category $(F\text{-})D_{\text{ovhol}}^b(X/K)$ is an analogue of the category of Weil complexes $(F\text{-})D_c^b(X \otimes_k \overline{k})$, and should not be regarded as an analogue of the derived category $D_c^b(X)$. In the ℓ -adic setting [BBD82, 5.4.5], the corresponding theorem to Theorem 4.3.6 is stated for complexes in $D_c^b(X)$, so there are slight differences in the formulation arising from the lack of the category corresponding to $D_c^b(X)$ in our theory. However, since there exists the factorization

$$D_c^b(X) \rightarrow F\text{-}D_c^b(X \otimes_k \overline{k}) \rightarrow D_c^b(X \otimes_k \overline{k}),$$

if the true derived category corresponding to $D_c^b(X)$ in the p -adic cohomology theory is constructed, Theorem 4.3.6 should not lose any information.

4.3.8. We conclude this paper by showing an application to ℓ -independence results of L -functions. To do this, we need to introduce, for a realizable variety X over a finite field k , the category $F\text{-}D_{\text{ovhol}}^b(X/\overline{\mathbb{Q}}_p)$. The construction is essentially the same as [AM11, 7.3]. Let L be a finite extension of the maximal unramified extension K^{ur} . An object of $F\text{-}D_{\text{ovhol}}^b(X/L)$ is a pair (\mathcal{E}, ρ) where $\mathcal{E} \in F\text{-}D_{\text{ovhol}}^b(X/K^{\text{ur}})$ and $\rho: L \rightarrow \text{End}_{F\text{-}D_{\text{ovhol}}^b(X/K^{\text{ur}})}(\mathcal{E})$ is a homomorphism of K^{ur} -algebras. By taking the cohomology functor \mathcal{H}_t^* , ρ induces a homomorphism $\mathcal{H}_t^*(\rho): L \rightarrow \text{End}_{F\text{-Ovhol}(X/K^{\text{ur}})}(\mathcal{H}_t^*(\mathcal{E}))$. The “heart” is denoted by $F\text{-Ovhol}(X/L)$.

For finite extensions $L'/L/K^{\text{ur}}$, there exists a scalar extension functor $F\text{-}D_{\text{ovhol}}^b(X/L) \rightarrow F\text{-}D_{\text{ovhol}}^b(X/L')$. This is “ t -exact” in the obvious sense, and induces a functor $F\text{-Ovhol}(X/L) \rightarrow F\text{-Ovhol}(X/L')$. By taking the limit over finite extensions L of K^{ur} , we have the categories $F\text{-}D_{\text{ovhol}}^b(X/\overline{\mathbb{Q}}_p)$ and $F\text{-Ovhol}(X/\overline{\mathbb{Q}}_p)$. We notice that $F\text{-Ovhol}(\text{Spec}(k)/\overline{\mathbb{Q}}_p)$ is nothing but the category of pairs (V, ϕ) where V is a finite dimensional $\overline{\mathbb{Q}}_p$ -vector space, and ϕ is an automorphism of V .

Since ι -weights remain to be the same after scalar extension, Theorem 4.1.3, Theorem 4.2.3, Corollary 4.2.4 remain to be true even after replacing K by $\overline{\mathbb{Q}}_p$.

Remark. Just to state Theorem 4.3.11, we do not need to introduce this generalized category. However, in the proof, we need to extend the scalar to $\overline{\mathbb{Q}}_p$, in order to produce sufficiently many sheaves \mathcal{L}_ρ using the notation of [Fuj02, §3].

4.3.9. Let X be a realizable variety over k . Let \mathcal{E} be an object in $F\text{-Ovhol}(X/\overline{\mathbb{Q}}_p)$. For a closed point x of X , let $\rho_x: \text{Spec}(k(x)) \rightarrow \text{Spec}(k)$ be the structural morphism. We define the *local L -factor at x* to be

$$L_x(\mathcal{E}, t) := \det_{\overline{\mathbb{Q}}_p} (1 - t^{\deg(x)} \cdot F; \rho_{x+} \circ i_x^+(\mathcal{E}))^{-1/\deg(x)},$$

and the *global L -function*⁽¹⁰⁾ to be

$$L(X, \mathcal{E}, t) := \prod_{x \in |X|} L_x(\mathcal{E}, t).$$

Let $K(X, \overline{\mathbb{Q}}_p)$ be the Grothendieck category of $F\text{-Ovhol}(X/\overline{\mathbb{Q}}_p)$. By additivity, the definition of local L -factor extends to a homomorphism

$$K(X, \overline{\mathbb{Q}}_p) \rightarrow \prod_{x \in |X|} (1 + t \cdot \overline{\mathbb{Q}}_p[[t]])^\times.$$

Definition 4.3.10. Let ℓ be a prime number different from p , and E be a field of characteristic 0. We fix embeddings $\overline{\mathbb{Q}}_p \hookleftarrow E \hookrightarrow \overline{\mathbb{Q}}_\ell$, where the first one is denoted by ι_p and the second by ι_ℓ . A pair $(\mathcal{E}, \mathcal{F}) \in K(X, \overline{\mathbb{Q}}_p) \times K(X, \overline{\mathbb{Q}}_\ell)$ is said to be an *E -compatible system* if for any closed point x of X , we have

$$\iota_p(L_x(\mathcal{E}, t)) = \iota_\ell(L_x(\mathcal{F}, t)) \in E[[t]].$$

Theorem 4.3.11 (Fujiwara-Gabber's ℓ -independence). *Compatible systems are stable under f_+ , $f^!$, \mathbb{D} , \otimes . Moreover, let $j: U \hookrightarrow X$ be an immersion of realizable varieties, and $(\mathcal{E}, \mathcal{F})$ be an E -compatible system such that \mathcal{E} (and hence \mathcal{F}) is ι_p -pure. Then $(j_{!+}(\mathcal{E}), j_{!*}(\mathcal{F}))$ is also an E -compatible system.*

Proof. The proof is exactly the same as [Fuj02]. In the proof, we need to use Lefschetz fixed point theorem. Since we are using “the extended scalar”, we need to check the theorem in this situation. For the verification, it is reduced to the overconvergent F -isocrystal case by the same argument as [Car06]. We repeat the dévissage argument of [ÉLS93, 6.2] to reduce to the case where the fixed point is empty, and in this case we repeat [ÉLS93, 5.4]. We leave the detailed verifications to the reader. ■

4.3.12. Let \mathcal{K} be a function field over k . We define

$$F\text{-Isoc}^\dagger(\mathcal{K}/K) := \varinjlim_U F\text{-Isoc}^\dagger(U/K)$$

where the inductive limit runs over smooth curves over k whose function field is \mathcal{K} .

Now, take E in $F\text{-Isoc}^\dagger(\mathcal{K}/K)$. By definition, there exists a smooth curve U whose function field is \mathcal{K} , and \tilde{E} be an overconvergent F -isocrystal on U . Let $j: U \hookrightarrow X$ be the smooth compactification. We define the *Hasse-Weil L -function* by

$$L_{\text{HW}}(\mathcal{K}, E, t) := L(X, j_{!+}(\text{sp}_+(\tilde{E})), t) \in E[[t]].$$

By definition, Hasse-Weil L -function does not depend on the choice of U . By Theorem 1.5.9 (iii), we may write the L -function as

$$L_{\text{HW}}(\mathcal{K}, E, t) = \prod_{x \in |X|} \det_K(1 - \varphi_x \cdot t^{\deg(x)}; \Psi_x(\tilde{\mathcal{E}})^{N=0, I=1})^{-1/\deg(x)},$$

where we used the notation of paragraph 2.3.3. In particular, our Hasse-Weil L -function is nothing but $L_{\text{WD}}(X, \tilde{E}(1), t)$ in [Mar08, 4.3.10] by Theorem 1.5.9 (i).

⁽¹⁰⁾ There are several ways of normalizations of global L -function. Ours is the same as that of [AM11, 7.2.3].

On the other hand, by Lefschetz trace formula and self-duality 1.4.3, we get the functional equation

$$L_{\text{HW}}(\mathcal{K}, E, t) = \varepsilon(\mathcal{K}, E) \cdot t^{g(\mathcal{K})-2} \cdot L_{\text{HW}}(\mathcal{K}, E^\vee, q \cdot t^{-1})$$

where $g(\mathcal{K})$ denotes the genus of the function field \mathcal{K} , and $\varepsilon(\mathcal{K}, E)$ is some number in $\overline{\mathbb{Q}}_p$ (cf. [AM11, 7.2.3]). This is a generalization of [Mar08, 4.3.11] to F -isocrystals of arbitrary ranks.

Now, let X be a realizable variety over \mathcal{K} . Then there exists a model $f: \mathcal{X} \rightarrow U$ where U is a smooth curve whose function field is \mathcal{K} , \mathcal{X} is a realizable variety, and the generic fiber of f is X . We denote by $p: \mathcal{X} \rightarrow \text{Spec}(k)$ the structural morphism. We define the cohomology $\mathcal{H}^m(X/K)$ in $F\text{-Isoc}^\dagger(\mathcal{K}/K)$ to be the one represented by $\mathcal{H}_t^m(f_+p^+(K))$. We can check easily that the construction does not depend on auxiliary choices. We end this paper with the following p -adic interpretation of Hasse-Weil L -function over function fields, which follows directly from Theorem 4.3.11:

Corollary 4.3.13. *Let $f: X \rightarrow \text{Spec}(\mathcal{K})$ be a realizable proper smooth scheme (e.g. projective smooth variety). Then we have*

$$L_{\text{HW}}(\mathcal{K}, H^m(\overline{X}), t) = L_{\text{HW}}(\mathcal{K}, \mathcal{H}^m(X/K), t),$$

where the left hand side is the Hasse-Weil L -function associated to the ℓ -adic representation $H^m(\overline{X}) := H^m(X \otimes_{\mathcal{K}} \mathcal{K}^{\text{sep}}, \mathbb{Q}_\ell)$ of $\text{Gal}(\mathcal{K}^{\text{sep}}/\mathcal{K})$.

A Internal homomorphism

In this appendix, we consider situation (A) in Notation and convention.

Definition A.1. Let \mathbb{Y} be a couple (1.1.2). Let $\mathcal{E}, \mathcal{F}, \mathcal{G} \in F\text{-}D_{\text{ovhol}}^b(\mathbb{Y}/K)$. We put

$$\mathcal{H}\text{om}_{\mathbb{Y}}(\mathcal{E}, \mathcal{F}) := \mathbb{D}_{\mathbb{Y}}(\mathcal{E}) \widetilde{\otimes}_{\mathbb{Y}} \mathcal{F}.$$

From the biduality isomorphism, we get the isomorphisms

$$\mathcal{H}\text{om}_{\mathbb{Y}}(\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} \mathcal{H}\text{om}_{\mathbb{Y}}(\mathbb{D}_{\mathbb{Y}}(\mathcal{F}), \mathbb{D}_{\mathbb{Y}}(\mathcal{E})), \quad \mathcal{H}\text{om}_{\mathbb{Y}}(\mathcal{E}, \mathcal{H}\text{om}_{\mathbb{Y}}(\mathcal{F}, \mathcal{G})) \xrightarrow{\sim} \mathcal{H}\text{om}_{\mathbb{Y}}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}). \quad (\text{A.1.1})$$

For any morphism $u: \mathbb{Y}' \rightarrow \mathbb{Y}$ of couples, by the biduality isomorphism and (1.1.9.1), we have

$$u^! \mathcal{H}\text{om}_{\mathbb{Y}}(\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} \mathcal{H}\text{om}_{\mathbb{Y}'}(u^+(\mathcal{E}), u^!(\mathcal{F})).$$

Lemma A.2. *Let $f: P \rightarrow S$ be a smooth morphism of noetherian schemes such that p is nilpotent in \mathcal{O}_S . For any $\mathcal{E}^{(m)} \in D_{\text{coh}}^b(\mathcal{D}_{P/S}^{(m)})$ and $\mathcal{F}^{(m)} \in D^b(\mathcal{D}_{P/S}^{(m)})$, we have the canonical isomorphism:*

$$f_+^{(m)}(\mathbb{D}_{P/S}^{(m)}(\mathcal{E}^{(m)}) \otimes_{\mathcal{O}_P}^{\mathbb{L}} \mathcal{F}^{(m)})[-d_P] \xrightarrow{\sim} \mathbb{R}f_*(\mathbb{R}\mathcal{H}\text{om}_{\mathcal{D}_{P/S}^{(m)}}(\mathcal{E}^{(m)}, \mathcal{F}^{(m)})).$$

Proof. This is the composition of the following isomorphisms:

$$\begin{aligned} f_+^{(m)}(\mathbb{D}_{P/S}^{(m)}(\mathcal{E}^{(m)}) \otimes_{\mathcal{O}_P}^{\mathbb{L}} \mathcal{F}^{(m)})[-d_P] &\xrightarrow{\sim} f_+^{(m)}(\mathbb{R}\mathcal{H}\text{om}_{\mathcal{D}_{P/S}^{(m)}}(\mathcal{E}^{(m)}, (\mathcal{D}_{P/S}^{(m)} \otimes_{\mathcal{O}_P} \omega_{P/S}^{-1}) \otimes_{\mathcal{O}_P}^{\mathbb{L}} \mathcal{F}^{(m)}))[-d_P]) \\ &= \mathbb{R}f_*(\omega_{P/S} \otimes_{\mathcal{D}_{P/S}^{(m)}}^{\mathbb{L}} \mathbb{R}\mathcal{H}\text{om}_{\mathcal{D}_{P/S}^{(m)}}(\mathcal{E}^{(m)}, (\mathcal{D}_{P/S}^{(m)} \otimes_{\mathcal{O}_P} \omega_{P/S}^{-1}) \otimes_{\mathcal{O}_P}^{\mathbb{L}} \mathcal{F}^{(m)})) \\ &\xrightarrow{\sim} \mathbb{R}f_{**}(\mathbb{R}\mathcal{H}\text{om}_{\mathcal{D}_{P/S}^{(m)}}(\mathcal{E}^{(m)}, \omega_{P/S} \otimes_{\mathcal{D}_{P/S}^{(m)}}^{\mathbb{L}} (\mathcal{D}_{P/S}^{(m)} \otimes_{\mathcal{O}_P} \omega_{P/S}^{-1}) \otimes_{\mathcal{O}_P}^{\mathbb{L}} \mathcal{F}^{(m)})) \\ &\xrightarrow{\sim} \mathbb{R}f_*(\mathbb{R}\mathcal{H}\text{om}_{\mathcal{D}_{P/S}^{(m)}}(\mathcal{E}^{(m)}, \mathcal{F}^{(m)})). \end{aligned}$$

where \star follows by [Car05, 2.1.26], and $\star\star$ follows by [Car05, 2.1.19]. ■

Definition A.3. Let \mathbb{Y} be a couple. Choose an l.p. frame $(Y, X, \mathcal{P}, \mathcal{Q})$ of \mathbb{Y} . Let $\mathcal{E}, \mathcal{F} \in D_{\text{ovhol}}^b(\mathbb{Y}/K)$ and $\mathcal{E}_{\mathcal{P}}, \mathcal{F}_{\mathcal{P}}$ be the corresponding objects in $D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$. Then, using Lemma 1.2.7, we notice that the complex of abelian groups $\mathbb{R}\text{Hom}_{D(\mathcal{D}_{\mathcal{P}, \mathcal{Q}}^\dagger)}(\mathcal{E}, \mathcal{F})$ does not depend on the choice of the l.p. frame $(Y, X, \mathcal{P}, \mathcal{Q})$ of \mathbb{Y} . We put

$$\mathbb{R}\text{Hom}_{D_{\text{ovhol}}^b(\mathbb{Y}/K)}(\mathcal{E}, \mathcal{F}) := \mathbb{R}\text{Hom}_{D(\mathcal{D}_{\mathcal{P}, \mathcal{Q}}^\dagger)}(\mathcal{E}_{\mathcal{P}}, \mathcal{F}_{\mathcal{P}}).$$

Remark A.4. Let \mathcal{E}, \mathcal{F} be objects of $\text{Ovhol}(\mathbb{Y}/K)$. Then we can check easily that

$$\mathcal{H}^1 \mathbb{R}\text{Hom}_{D_{\text{ovhol}}^b(\mathbb{Y}/K)}(\mathcal{E}, \mathcal{F}) = \text{Ext}_{D_{\mathbb{Y}/K}^\dagger}^1(\mathcal{E}, \mathcal{F}),$$

where $\text{Ext}_{D_{\mathbb{Y}/K}^\dagger}^1(\mathcal{E}, \mathcal{F})$ denotes the Yoneda's Ext group in the abelian category $\text{Ovhol}(\mathbb{Y}/K)$.

Proposition A.5. Let \mathbb{Y} be a couple and $a: \mathbb{Y} \rightarrow \text{Spec}(k)$ the structural morphism which we assume to be complete. For any $\mathcal{E}, \mathcal{F} \in F\text{-}D_{\text{ovhol}}^b(\mathbb{Y}/K)$, we have the isomorphism

$$a_+ \mathcal{H}\text{om}_{\mathbb{Y}}(\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} \mathbb{R}\text{Hom}_{D_{\text{ovhol}}^b(\mathbb{Y}/K)}(\mathcal{E}, \mathcal{F}).$$

Proof. Choose an l.p. frame $(Y, X, \mathcal{P}, \mathcal{Q})$ of \mathbb{Y} . Let $\mathcal{E}, \mathcal{F} \in F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$. Let f be the structural morphism of \mathcal{P} and $Z := X \setminus Y$. By definition, we get:

$$\begin{aligned} a_+ \mathcal{H}\text{om}_{\mathbb{Y}}(\mathcal{E}, \mathcal{F}) &\cong f_+((\dagger Z) \circ \mathbb{D}_{\mathcal{P}}(\mathcal{E}) \otimes_{\mathcal{O}_{\mathcal{P}, \mathcal{Q}}}^\mathbb{L} \mathcal{F})[-d_P] \xrightarrow{\sim} f_+(\mathbb{D}_{\mathcal{P}}(\mathcal{E}) \otimes_{\mathcal{O}_{\mathcal{P}, \mathcal{Q}}}^\mathbb{L} \mathcal{F})[-d_P], \\ \mathbb{R}\text{Hom}_{D_{\text{ovhol}}^b(\mathbb{Y}/K)}(\mathcal{E}, \mathcal{F}) &\cong \mathbb{R}\text{Hom}_{D(\mathcal{D}_{\mathcal{P}, \mathcal{Q}}^\dagger)}(\mathcal{E}, \mathcal{F}). \end{aligned}$$

Thus, we are reduced to the case where $Y = X = P$. Let $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in \underline{LD}_{\mathbb{Q}, \text{coh}}^b(\widehat{\mathcal{D}}_{\mathcal{P}}^{(\bullet)})$ such that $\varinjlim \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}$ and $\varinjlim \mathcal{F}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}$. We can suppose that there exists an increasing map $\lambda: \mathbb{N} \rightarrow \mathbb{N}$ such that for any integers $m' \geq m$ we have $\lambda(m) \geq m$, $\mathcal{E}^{(m)} \in D_{\text{coh}}^b(\widehat{\mathcal{D}}_{\mathcal{P}}^{(\lambda(m))})$ and the canonical morphism

$$\widehat{\mathcal{D}}_{\mathcal{P}}^{(\lambda(m'))} \otimes_{\widehat{\mathcal{D}}_{\mathcal{P}}^{(\lambda(m))}}^\mathbb{L} \mathcal{E}^{(m)} \rightarrow \mathcal{E}^{(m')}$$

is an isomorphism. By taking projective limits, we deduce from Lemma A.2 the isomorphism

$$f_+^{(\lambda(m))}(\mathbb{D}^{\lambda(m)}(\mathcal{E}^{(m)}) \otimes_{\mathcal{O}_{\mathcal{P}}}^\mathbb{L} \mathcal{F}^{(\lambda(m))})[-d_P] \xrightarrow{\sim} \mathbb{R}\text{Hom}_{D(\widehat{\mathcal{D}}_{\mathcal{P}}^{(\lambda(m))})}(\mathcal{E}^{(m)}, \mathcal{F}^{(\lambda(m))}). \quad (\star)$$

Since $\mathcal{E}^{(m)} \in D_{\text{coh}}^b(\widehat{\mathcal{D}}_{\mathcal{P}}^{(\lambda(m))})$, we get the isomorphisms $\widehat{\mathcal{D}}_{\mathcal{P}}^{(\lambda(m'))} \otimes_{\widehat{\mathcal{D}}_{\mathcal{P}}^{(\lambda(m))}}^\mathbb{L} \mathbb{D}^{\lambda(m)}(\mathcal{E}^{(m)}) \rightarrow \mathbb{D}^{\lambda(m')}(\mathcal{E}^{(m')})$ and $\widehat{\mathcal{D}}_{\mathcal{P}}^{(\lambda(m'))} \otimes_{\widehat{\mathcal{D}}_{\mathcal{P}}^{(\lambda(m))}}^\mathbb{L} \mathbb{R}\text{Hom}_{D(\widehat{\mathcal{D}}_{\mathcal{P}}^{(\lambda(m))})}(\mathcal{E}^{(m)}, \mathcal{F}^{(\lambda(m'))}) \xrightarrow{\sim} \mathbb{R}\text{Hom}_{D(\widehat{\mathcal{D}}_{\mathcal{P}}^{(\lambda(m'))})}(\mathcal{E}^{(m')}, \mathcal{F}^{(\lambda(m'))})$. By taking the direct limit to (\star) and tensoring by \mathbb{Q} , we get the desired isomorphism. ■

Proposition A.6 (Projection formulas). Let $u: \mathbb{Y} \rightarrow \mathbb{Y}'$ a complete morphism of couples. For any $\mathcal{E} \in F\text{-}D_{\text{ovhol}}^b(\mathbb{Y}/K)$, $\mathcal{E}' \in F\text{-}D_{\text{ovhol}}^b(\mathbb{Y}'/K)$,

$$u_+(\mathcal{E} \widetilde{\otimes} u^!(\mathcal{E}')) \xrightarrow{\sim} u_+(\mathcal{E}) \widetilde{\otimes} \mathcal{E}', \quad u_!(\mathcal{E} \otimes u^+(\mathcal{E}')) \xrightarrow{\sim} u_!(\mathcal{E}) \otimes \mathcal{E}'. \quad (\text{A.6.1})$$

Proof. Let $u = (\star, \star, g, \star): (Y, X, \mathcal{P}, \mathcal{Q}) \rightarrow (Y', X', \mathcal{P}', \mathcal{Q}')$ be a morphism of frames for u . We put $Z := X \setminus Y$ and $Z' := X' \setminus Y'$. Let $\mathcal{E} \in F\text{-}D_{\text{ovhol}}^b(Y, \mathcal{P}/K)$ and $\mathcal{E}' \in F\text{-}D_{\text{ovhol}}^b(Y', \mathcal{P}'/K)$. Then we have

$$u_+(\mathcal{E} \widetilde{\otimes} u^!(\mathcal{E}')) \xrightarrow{\sim} g_+(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{P}, \mathcal{Q}}}^\mathbb{L} g^!(\mathcal{E}'))[-d_P], \quad u_+(\mathcal{E}) \widetilde{\otimes} \mathcal{E}' \xrightarrow{\sim} g_+(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{P}', \mathcal{Q}'}}^\mathbb{L} \mathcal{E}')[-d_{P'}].$$

Hence, from the projection formula [Car04, 2.1.4], we get the first one. By using the biduality isomorphism, this implies the second one. ■

Proposition A.7 (Adjointness Properties). *Let $u: \mathbb{Y} \rightarrow \mathbb{Y}'$ a morphism of couples. For any $\mathcal{E} \in F-D_{\text{ovhol}}^b(\mathbb{Y}/K)$, $\mathcal{E}' \in F-D_{\text{ovhol}}^b(\mathbb{Y}'/K)$, we get the functorial isomorphism:*

$$u_+ \mathcal{H}om_{\mathbb{Y}}(\mathcal{E}, u^!(\mathcal{E}')) \xrightarrow{\sim} \mathcal{H}om_{\mathbb{Y}'}(u_!(\mathcal{E}), \mathcal{E}').$$

Proof. We have

$$u_+ \mathcal{H}om_{\mathbb{Y}}(\mathcal{E}, u^!(\mathcal{E}')) = u_+(\mathbb{D}_{\mathbb{Y}}(\mathcal{E}) \widetilde{\otimes} u^!(\mathcal{E}')) \xrightarrow[\star]{\sim} (u_+ \circ \mathbb{D}_{\mathbb{Y}})(\mathcal{E}) \widetilde{\otimes} \mathcal{E}' \xrightarrow{\sim} \mathcal{H}om_{\mathbb{Y}'}(u_!(\mathcal{E}), \mathcal{E}').$$

where \star follows by (A.6.1). ■

Proposition A.8. *Let $u: \mathbb{Y} \rightarrow \mathbb{Y}'$ a morphism of couples. For any $\mathcal{E} \in F-D_{\text{ovhol}}^b(\mathbb{Y}/K)$, $\mathcal{E}' \in F-D_{\text{ovhol}}^b(\mathbb{Y}'/K)$, we get the functorial isomorphism:*

$$u_+ \mathcal{H}om_{\mathbb{Y}}(u^+(\mathcal{E}'), \mathcal{E}) \xrightarrow{\sim} \mathcal{H}om_{\mathbb{Y}'}(\mathcal{E}', u_+(\mathcal{E})).$$

Proof. This is a consequence of (A.1.1) and Proposition A.7. ■

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